Answers to Formulate Mathematical Conditions...

Question 1 Obviously, if $a, b > 0$ and $a + b = 1$, then all four outer rectangles are of the same shape.

Here is the first, longer, and more involved solution: We deduce the side lengths of R_2, R_3 and R_4, assuming they all have area ab. This leads to

- R_2 has side lengths $1 - b$, $\frac{ab}{1-b}$.
- R_3 has side lengths $\frac{1-b-ab}{1-b}$, $\frac{ab-ab^2}{1-b-ab}$.
- R_4 has side lengths $\frac{1-b-2ab+ab^2}{1-b-ab}$, $ab\frac{1-b-ab}{1-b-2ab+ab^2}$.

Since the left hand sides of R_1 and R_4 must add up to 1, we obtain the condition

$$ab\frac{1-b-ab}{1-b-2ab+ab^2} + a = 1.$$

Multiplying both sides with $1 - b - 2ab + ab^2$, we obtain

$$\begin{aligned}
ab(1-b-ab) + a(1-b-2ab+ab^2) &= 1 - b - 2ab + ab^2, \\
ab - ab^2 - a^2b^2 + a - ab - 2a^2b + a^2b^2 &= 1 - b - 2ab + ab^2, \\
a + b - 2ab^2 - 2a^2b + 2ab - 1 = 0, & \\
(a + b - 1)(1 - 2ab) = 0. &
\end{aligned}$$

This shows that we have necessarily $a + b = 1$ or $ab = \frac{1}{2}$. The latter cannot be, since then each of the four rectangles would have area $\frac{1}{2}$, adding up to 2, which would exceed the area of the square with area 1 in which they are contained. Therefore, for geometric reasons, we must have $a + b = 1$. **Note that this solution is a direct proof!**

Here is the second, elegant and very short solution: Assume that $a + b \neq 1$. In the case that $a + b > 1$, the horizontal side of R_2 is strictly smaller than a and the vertical side of R_2 strictly bigger than b. This implies that the vertical side of R_3 is strictly smaller than a and the horizontal side of R_3 is strictly bigger than b. This implies that the horizontal side of R_4 is strictly smaller than a and the vertical side of R_4 is strictly bigger than b. But then the vertical side of R_1 must be strictly smaller than a, which is a contradiction. Assuming $a + b < 1$, a similar reasoning leads, again, to a contradiction. This proves $a + b = 1$ without a single calculation. **Note that this solution is an indirect proof: We start with the negation of the statement and show that this leads to a contradiction!**

Question 2 We start with the following three entries $a, b, c \in \mathbb{R}$ of a magic square M and calculate the other entries using the sum conditions:

$$M = \begin{pmatrix} a & b & * \\ c & * & * \\ * & * & * \end{pmatrix}.$$

Using the sum condition for the first row and to the first column, we obtain

$$M = \begin{pmatrix} a & b & -a-b \\ c & * & * \\ -a-c & * & * \end{pmatrix}.$$

Using now the sum condition for one of the two diagonals leads to

$$M = \begin{pmatrix} a & b & -a-b \\ c & 2a+b+c & * \\ -a-c & * & * \end{pmatrix}.$$

Using now the sum condition for the second row and second column leads to

$$M = \begin{pmatrix} a & b & -a-b \\ c & 2a+b+c & -2a-b-2c \\ -a-c & -2a-2b-c & * \end{pmatrix}.$$

The sum condition for the third column or the third row leads to the same final entry, namely $3a+2b+2c$:

$$M = \begin{pmatrix} a & b & -a-b \\ c & 2a+b+c & -2a-b-2c \\ -a-c & -2a-2b-c & 3a+2b+2c \end{pmatrix}.$$

Now, all sum conditions are satisfied, except for the sum along the main diagonal. This leads to the condition

$$6a + 3b + 3c = 0,$$

or, equivalently $2a+b+c = 0$. Therefore, the central entry of M must vanish:

$$M = \begin{pmatrix} a & b & -a-b \\ c & 0 & -2a-b-2c \\ -a-c & -2a-2b-c & 3a+2b+2c \end{pmatrix}.$$

Now, using again the sum condition for the second row and second column leads to the following simplification

$$M = \begin{pmatrix} a & b & -a-b \\ c & 0 & -c \\ -a-c & -b & 3a+2b+2c \end{pmatrix}.$$

Finally, using the sum condition for the main diagonal yields

$$M = \begin{pmatrix} a & b & -a-b \\ c & 0 & -c \\ -a-c & -b & -a \end{pmatrix}.$$

We conclude that every magic square is of this form, under the additional condition $2a + b + c = 0$. This allows us to answer both questions:

(a) The entries of the first row determine uniquely the whole magic square, since they pin down the variables a and b, and therefore also the variable $c = -2a - b$, and therefore also the whole magic square.

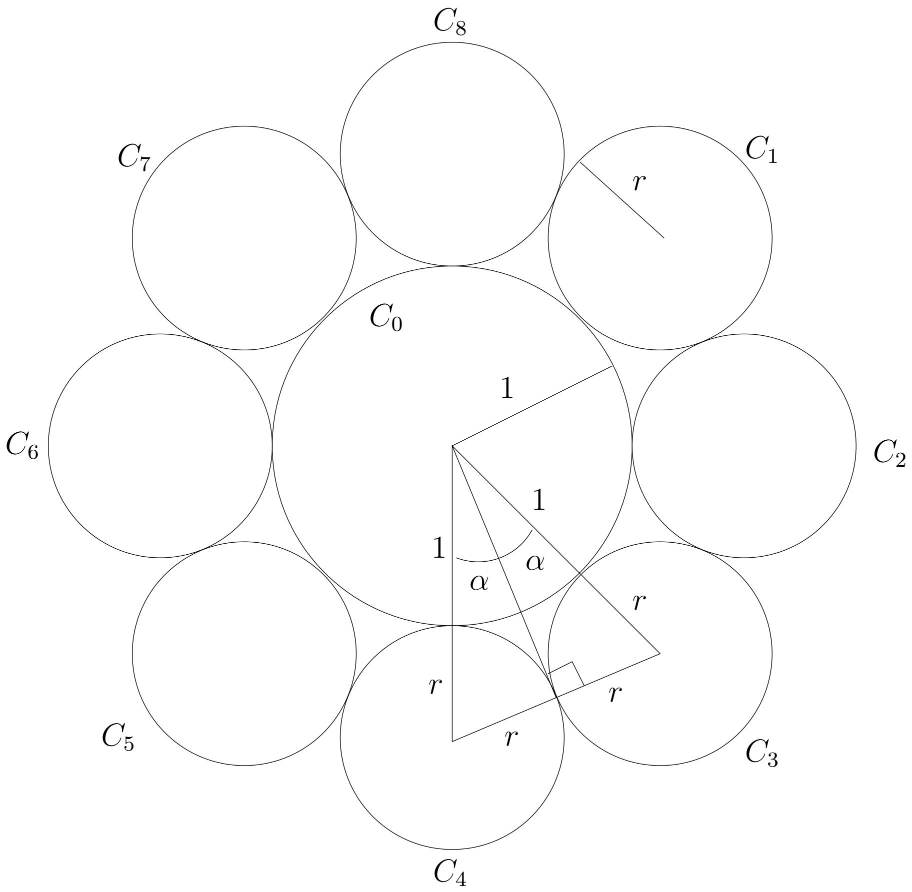
(b) The entries of the second row determine only the value of c and there are infinitely many solutions $a, b \in \mathbb{R}$ for $c = -2a - b$. One solution is $(a, b) = (0, -c)$, leading to the magic square

$$M = \begin{pmatrix} 0 & -c & c \\ c & 0 & -c \\ -c & c & 0 \end{pmatrix},$$

and another solution is $(a, b) = (-c, c)$, leading to the magic square

$$M = \begin{pmatrix} -c & c & 0 \\ c & 0 & -c \\ 0 & -c & c \end{pmatrix}.$$

Question 3 Here is an illustration of the configuration with the right radius $r > 0$.



We have obviously $\alpha = \frac{2\pi}{16} = \frac{\pi}{8}$ and

$$\sin\alpha = \frac{r}{1+r},$$

i.e.,

$$r = \frac{\sin\alpha}{1-\sin\alpha} \approx \frac{0.3826834}{0.6173166} \approx 0.6199.$$