Number Problems (Week 8)

Question 1 Prove the following remarkable fact, which holds for every three digits number N with decreasing digits (say, $N = 863$).

- Reverse the digits of N and call the new number M.
- Subtract M from N and call the new number A.
- Reverse the digits of A and call the new number B.
- Calculate $A + B$. You always end up with 1089.

Question 2 Analyse the following game: There are 100 pieces of paper in a hat, carrying the numbers 1 to 100. You draw two pieces of paper from the hat and subtract the smaller number from the larger one. Then you write this difference on a new piece of paper and throw this back into the hat, discarding the original two pieces of paper. The hat now contains 99 pieces of paper. You repeat this procedure 99 times until there is only one piece of paper left in the hat.

Question 3 Consider the number $A = 100!$ in the decimal representation.

(a) How many zero digits are there at the end?

(b) Using only elementary arguments and no calculator, try to find as good as possible exponents $a < b$ such that $10^a < A < 10^b$.

(c) Show that

$$100 \log(100) - 99 \leq \log(100!) \leq 101 \log(101) - 100,$$

where log denotes the natural logarithm.