Preimages and Equivalence Relations (Week 9)

Question 1 Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 + y^2 + z^2$. Describe the preimages $f^{-1}(\{-1\})$, $f^{-1}(\{0\})$, $f^{-1}(\{1\})$ and $f^{-1}([1,2])$ geometrically.

Question 2 Let $f : [0,4] \to \mathbb{R}$, $f(x) = \sin(\pi x)$. Sketch the graph of f and determine the preimage $f^{-1}([0,1)) \subset \mathbb{R}$.

Question 3 Let $f : \mathbb{R} \to \mathbb{R}$ be a map. As introduced in the lecture, the image $f(X)$ of a set $X \subset \mathbb{R}$ is defined as

$$f(X) := \{f(x) \mid x \in X\},$$

that is, all image points of set X under f. Moreover, the preimage $f^{-1}(X)$ of a set $Y \subset \mathbb{R}$ is defined as

$$f^{-1}(Y) := \{x \in X \mid f(x) \in Y\},$$

that is, all points which are mapped into Y via f.

(a) Let $Y_1, Y_2 \subset \mathbb{R}$. Show that

$$f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2).$$

(b) Let $f(x) = x^2$. Find two sets $X_1, X_2 \subset \mathbb{R}$ such that

$$f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2).$$

This shows that property (a) for preimages does not hold for images.

Question 4 Let $f : X \to Y$ and $g : Y \to Z$ be maps. Show the following facts:

(a) If f and g are injective, then $g \circ f$ is also injective.

(b) If f and g are surjective, then $g \circ f$ is also surjective.

(c) If f and g are bijective, then $g \circ f$ is also bijective and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Question 5 Check whether the following definitions are equivalence relations:

(a) On $\mathbb{R}$: $x \sim y$ if $xy = 0$.

(b) On $\mathbb{R}$: $x \sim y$ if $x - y \in \mathbb{Q}$.

(c) On $\mathbb{R}^2$: $(x, y) \sim (x', y')$ if $x^2 - (x')^2 = y^2 - (y')^2$.

(d) On $\mathbb{R}^2$: $(x, y) \sim (x', y')$ if $(x, y) \perp (x', y')$.

(e) On $\mathbb{R}^n$, $n \geq 2$: $v \sim w$ if v and w are linearly dependent.

(f) On real $n \times n$ matrices: $A \sim B$ if there exists an invertible real matrix X such that $A = XBX^{-1}$.

(g) On finite and infinite sets: $X \sim Y$ if X and Y have the same cardinality.

(h) On continuous functions $f, g : [0, 1] \to \mathbb{R}$: $f \sim g$ if $\int_0^1 f(x) - g(x) dx = 0$.

Question 6 Show that the following definition is an equivalence relation on $\mathbb{N} \times \mathbb{N}$:

$$(a, b) \sim (c, d) \qquad \Leftrightarrow \qquad ad = bc.$$

We denote the equivalence class of (a, b) by $[a, b]$. Show that

$$[a, b] \otimes [c, d] := [ac, ad - bc]$$

is a well-defined operation on the equivalence classes.

Question 7 Let $\mathbb{R}[x]$ be the set of all real polynomials. Show that the following definition is an equivalence relation on $\mathbb{R}[x]$:

$$p(x) \sim q(x) \qquad \Leftrightarrow \qquad p(x) - q(x) \text{ is divisible by } x^2 + 1.$$

We denote the equivalence class of $p(x)$ by $[p(x)]$.

(a) Show that

$$[(x^2 + 7)(x - 3)] = [6x - 18].$$

(b) Show that every equivalence class $[p(x)]$ has a representative of the form $ax + b$ with $a, b \in \mathbb{R}$, i.e., $p(x) \sim ax + b$.

(c) Show that the map

$$[ax + b] \mapsto ai + b$$

is a bijection between the equivalence classes of $\mathbb{R}[x]$ and the complex numbers $\mathbb{C}$ and that

$$[(ax + b)(cx + d)] \mapsto (ai + b)(ci + d).$$