Answers to Potpourri of Problems

Question 1 The "proof" starts with the statement to be proved and derives from it, via a sequence of implications, a true statement under the given assumption "$u_n > a$". This means that the arguments show, under the assumption of $u_n > a$, that the assumption $u_{n+1} > a$ leads to a true statement. Let us analyse this in detail: Let A denote the statement "$u_n > a$" and B denote the statement "$u_{n+1} > a$". So the arguments show that A and B together imply the true statement $(u_n - a)^2 > 0$", i.e.,

$$(A \text{ and } B) \Rightarrow \text{True}. \tag{1}$$

Assuming A to be true, this does not mean anything for the validity of B: If B is false and "A and B" is therefore also false, (1) is then

$$\text{False} \Rightarrow \text{True},$$

which is, again, a true statement. So the statement B can be false or true and the arguments prove nothing.

Question 2 All equalities are correct, except for

$$\{x^2 \mid x \leq -1 \text{ and } x > 2\} = \{x^2 \mid x \leq -1\} \cap \{x^2 \mid x > 2\}.$$

The problem here is the following: While the identity

$$\{x \mid x \text{ has property A and property B}\} = \\ \{x \mid x \text{ has property A}\} \cap \{x \mid x \text{ has property B}\}$$

is correct, we only have

$$\{f(x) \mid x \text{ has property A and property B}\} \subset \\ \{f(x) \mid x \text{ has property A}\} \cap \{f(x) \mid x \text{ has property B}\}.$$

So, for example, if the properties A and B are mutually exclusive then the set on the left side is empty. But the function f might map two different elements x_1 and x_2 with the incompatible properties A and B, respectively, to the same element $y = f(x_1) = f(x_2)$, in which case the set on the right side is not empty and contains, at least, the element y. So we see that the equality need not be correct if f is not injective, and an example where exactly this is exploited, is given in the question.

Question 3

(a) If a triangle is not right–angled then its side lengths a, b, c do not satisfy $a^2 + b^2 = c^2$.

(b) If a sequence (x_n) of real numbers is not convergent then it is (**not** monotone increasing) **or** (**not** bounded from above).
Here it is important to know De Morgan's Rule: The negation of "A and B" is "(not A) or (not B)". Also, you have to be aware that "C or D" is not exclusive, that is, this statement is also true if both C and D are true.

(c) If a sequence $(x_n)_{n=1}^{\infty}$ of real numbers has a subsequence that does not converge to a limit x_∞ then the sequence (x_n) itself does also not converge to x_∞.
Here it is important to know that the negation swaps "for all" and "there exists".

(d) If there is a pair of opposite angles of a quadrilateral in the plane which do not add up to 180^o then its four vertices do not lie on a common circle.

Question 4

(a) Let A and B two finite sets. For a map $f : A \to B$ to be bijective it is **necessary** that $|A| = |B|$.

Explanation: If f is bijective then there is a one-one correspondende between the elements of A and B via the map f. This means that both sets must have the same cardinality. The condition is not sufficient since $A = B = \{1, 2\}$, $f(1) = f(2) = 1$ satisfies $|A| = |B|$ but is not injective.

(b) Let (x_n) be a sequence of non-negative real numbers. For $(-1)^n x_n$ to be convergent it is **necessary and sufficient** that $x_n \to 0$.

Explanation: Since $(-1)^n x_n$ is an alternating sequence, its limit must be 0 if it is convergent. But we obviously have $(-1)^n x_n \to 0$ if $x_n \to 0$, and vice versa.

(c) For a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous it is **sufficient** that f is differentiable.

Explanation: It is known from school that every differentiable function is continuous. But not every continuous function must be differentiable: a prominent example is the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$.

(d) Let (a_n) be a sequence of real numbers. For a series $\sum_{n=1}^{\infty} a_n$ to be convergent it is **necessary** that $a_n \to 0$.

Explanation: Let $A_k = \sum_{n=1}^{k} a_n$. Since (A_k) is convergent, we have $\lim A_k = c$ for some $c \in \mathbb{R}$. Since $a_n = A_{n+1} - A_n$, we have $\lim a_n = \lim A_{n+1} - A_n = c - c = 0$. This shows that the condition is necessary. But the condition is not sufficient since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent but we have $\frac{1}{n} \to 0$.

(e) For two vectors $v_1, v_2 \in \mathbb{R}^2$ to be linearly independent it is **sufficient** that they are non-zero and orthogonal to each other.

Explanation: If two vectors v_1, v_2 are non-zero and orthogonal to each other, we have $v_1 \cdot v_1 \neq 0 \neq v_2 \cdot v_2$ and $v_1 \cdot v_2 = 0$. If $av_1 + bv_2 = 0$, we obtain by multiplication with v_1 that $a = 0$ and by multiplication with v_2 that $b = 0$. Therefore, v_1, v_2 are linearly independent. But this condition is not necessary: $v_1 = (1,0)$ and $v_2 = (1,1)$ are linearly independent but we have $v_1 \cdot v_2 = 1$, that is, v_1 and v_2 are not orthogonal to each other.

(f) Let A, B be two 2×2 real matrices and Id the 2×2 identity matrix. For $AB = BA$ it is **sufficient** that one the two matrices is a multiple of Id.

Explanation: If $A = a \cdot \mathrm{Id}$, we have $AB = aB = BA$. Similarly, if $B = b \cdot \mathrm{Id}$, we have $AB = bA = BA$. But this condition is not necessary. For example, we have $AB = BA$ for the matrices $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$.

Question 5

(a) You ask all guests in the hotel to move five rooms, i.e., the guest in room k moves to room $k+5$. Then the rooms $1, 2, 3, 4, 5$ are emptied, and you can assign them to the five new guests.

(b) You ask all guests to change from their room k into the room $2k$. This empties all rooms with odd numbers. You assign the room $2j-1$ to the j-th new guest.

(c) You ask the passengers of bus $i \in \mathbb{N}$ to form the i-th countably infinite row. Then you enumerate the guests in a diagonal manner (as shown in Lecture 9 to prove that the set of positive rationals is countable) and assign to them the corresponding room with the same number.

Question 6

(a) We know that $\mathbb{Q}$ is countable. Another diagonal argument shows that $\mathbb{Q}^2$ is also countable. Let

$$\mathbb{Q}^2 = \{q_1, q_2, q_3, \dots\}$$

be an enumeration of $\mathbb{Q}^2$. Let $\mathcal{D}$ be a set of discs as described in the question. Then we find for every disc $D \in \mathcal{D}$ a point q_j with smallest index j such that $q_j \in D$ (since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$). The disjointness of the discs guarantees that there are no two discs D, D' with the same associated points $q, q' \in \mathbb{Q}^2$. This shows that we have a bijection between the elements of $\mathcal{D}$ and a subset of $\mathbb{Q}^2$. This shows that $\mathcal{D}$ must be countable.

(b) Note first that a convergent sequence (x_n) of natural numbers must be constant from some finite index $n_0 \in \mathbb{N}$ onwards. Let $S_{N,k}$ be set of all sequences (x_n) of natural numbers with $x_n = k$ for all $n \geq N$ and $|x_n| \leq N$ for all $n \in \mathbb{N}$. It then becomes clear that the set S of all convergent sequences (x_n) whose elements are natural numbers is the union $\bigcup_{N,k \in \mathbb{N}} S_{N,k}$. This is a countable union of countable sets, i.e., itself again countable.

(c) Assume that the set T of all rational sequences (x_n) with $\lim x_n = 0$ is countable. We can then enumerate the elements of T, i.e.,

$$T = \{(x_{1,n}), (x_{2,n}), \dots\}.$$

We now construct the following sequence (y_n): We set $y_n = 1/n \in \mathbb{Q}$ if $x_{n,n} \neq 1/n$ and $y_n = 1/(n+1) \in \mathbb{Q}$, otherwise. We obviously have $(y_n) \in T$, but (y_n) does not agree with any of the sequences $(x_{i,n})$, $i \in \mathbb{N}$. This is a contradiction.