Dr. Norbert Peyerimhoff, Durham University 16/10/2008

Riemannian Geometry IV

Problems, set 1.

Exercise 1. The real projective space $\mathbb{R}P^n$ consists of all lines through the origin in $\mathbb{R}^{n+1}$. In this exercise, we equip it with a differentiable atlas. This is the main step for the proof that $\mathbb{R}P^n$ carries a manifold structure. For simplicity, we restrict to the case $n = 2$, the general case works completely analogous.

There are three coordinate charts $\varphi_j : U_j \to \mathbb{R}^2$ of $\mathbb{R}P^2$, defined as follows:

$$U_j := \{\mathbb{R} \cdot (p_1, p_2, p_3) \mid p_j \neq 0\},$$

$1 \leq j \leq 3$, and

$$\varphi_1(\mathbb{R} \cdot (p_1, p_2, p_3)) := (\frac{p_2}{p_1}, \frac{p_3}{p_1}), \quad \varphi_2(\mathbb{R} \cdot (p_1, p_2, p_3)) := (\frac{p_1}{p_2}, \frac{p_3}{p_2}),$$
$$\varphi_3(\mathbb{R} \cdot (p_1, p_2, p_3)) := (\frac{p_1}{p_3}, \frac{p_2}{p_3}).$$

Explain why U_1, U_2, U_3 cover all of $\mathbb{R}P^2$. Give the formulas for the inverse maps φ_j^{-1}. Determine the domain and the range of the coordinate changes $\varphi_i \circ \varphi_j^{-1}$ and calculate the coordinate changes explicitely. Explain why they are differentiable maps.

Exercise 2. This exercise shows that the matrix group $SL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det A = 1\}$ is a differentiable manifold, using Theorem 1.5.

(a) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Prove *Euler's relation*

$$\langle \operatorname{grad} f(x), x \rangle = m f(x),$$

where

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right).$$

Hint: Differentiate $\lambda \mapsto f(\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$ with respect to λ and use homogeneity.

(b) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Show that every value $y \neq 0$ is a regular value of f.

(c) Use the fact that $\det A$ is a homogeneous polynomial in the entries of A in order to show that $SL(n, \mathbb{R})$ is a differentiable manifold.