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Riemannian Geometry IV

Problems, set 2 (to be handed in on 31 October 2008).

Exercise 3. Let M be a differentiable manifold, $U \subset M$ open and $\varphi = (x_1, \dots, x_n) : U \to V_1 \subset \mathbb{R}^n$, $\psi = (y_1, \dots, y_n) : U \to V_2 \subset \mathbb{R}^n$ be two coordinate charts. Show for $p \in U$:

$$\frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^{n} \frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p,$$

where $y_j \circ \varphi^{-1} : V_1 \to \mathbb{R}$ and $\frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}$ is the classical partial derivative in the coordinate direction x_i of $\mathbb{R}^n$. **Hint:** Blow $f \circ \varphi^{-1}$ up to the expression $f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$, and apply the chain rule.

Exercise 4. Prove that the tangent space of the Lie group $SO(n) \subset M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ at the identity I is given by

$$T_I SO(n) = \{A \in M(n, \mathbb{R}) \mid A^\top = -A\},$$

i.e., the space of all skew-symmetric $n \times n$-matrices. **Hint:** You may use that we have, componentwise, $(AB)'(t) = A'(t)B(t) + A(t)B'(t)$, for the product of any two matrix-valued curves.