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Riemannian Geometry IV

Problems, set 3.

Exercise 5. Let X, Y be two vector fields on $\mathbb{R}^3$ defined by

$$\begin{aligned} X(x_1, x_2, x_3) &= (2x_3 - x_2)\frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3}, \\ Y(x_1, x_2, x_3) &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}. \end{aligned}$$

(a) Calculate the Lie bracket $[X, Y]$.

(b) Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the standard unit sphere. Show that the restrictions of the vector fields X, Y to S^2 are vector fields on S^2. **Hint:** You just have to show that, for every $x \in S^2$, $X(x)$ and $Y(x)$ lie in $T_x S^2$.

(c) Check that the restriction of the Lie bracket $[X, Y]$ to S^2 is also a vector field on S^2.

Additional remark: In fact, the following general result is true (you don't have to prove it): Let X, Y be two vector fields on $\mathbb{R}^n$ and $M \subset \mathbb{R}^n$ be a differentiable submanifold. Assume that the restrictions of X, Y to M are again vector fields on M. Then the Lie bracket of these restrictions within M coincides with the restriction of the Lie bracket of the original vector fields X, Y in $\mathbb{R}^n$ to M.

Exercise 6. Show the following properties (a)-(c) of the Lie bracket:

(a) $[X, Y] = -[Y, X]$.

(b) $[aX + bY, Z] = a[X, z] + b[Y, Z]$ for $a, b \in \mathbb{R}$.

(c) *Jacobi's identity* for Lie brackets:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Note that the webpage for Riemannian Geometry IV is at

`http://www.maths.dur.ac.uk/~dma0np/riemgeom0809/riemgeom.html`