Dr. Norbert Peyerimhoff, Durham University 14/11/2008

Riemannian Geometry IV

Problems, set 5 (Exercises 9 and 11 to be handed in on 28 November 2008).

Exercise 9. An almost global coordinate chart of the torus of revolution $T^2 \subset \mathbb{R}^3$ is given by $\varphi : U \to V = (0, 2\pi) \times (0, 2\pi) \subset \mathbb{R}^2$,

$$\varphi^{-1}(x_1, x_2) = ((R + r \cos x_1) \cos x_2, (R + r \cos x_1) \sin x_2, r \sin x_1),$$

with $r > 0$ and $R > r$. Explain the precise geometric shape which is rotated around the vertical axis in $\mathbb{R}^3$ in order to obtain T^2. Show that

$$\mathrm{vol}(T^2) = (2\pi r)(2\pi R) = 4\pi^2 r R.$$

Exercise 10. Use the same coordinate chart of the torus T^2 as in Exercise 10 and derive (with methods from Differential Geometry) that the Gaussian curvature $K : T^2 \to \mathbb{R}$ is given by

$$K(\varphi^{-1}(x_1, x_2)) = \frac{\cos x_1}{r(R + r \cos x_1)}.$$

Explain geometrically why

$$K(\varphi^{-1}(\pi/2, x_2)) = K(\varphi^{-1}(3\pi/2, x_2)) = 0 \quad \text{and}\, K(\varphi^{-1}(\pi, x_2)) < 0.$$

Show by calculation that

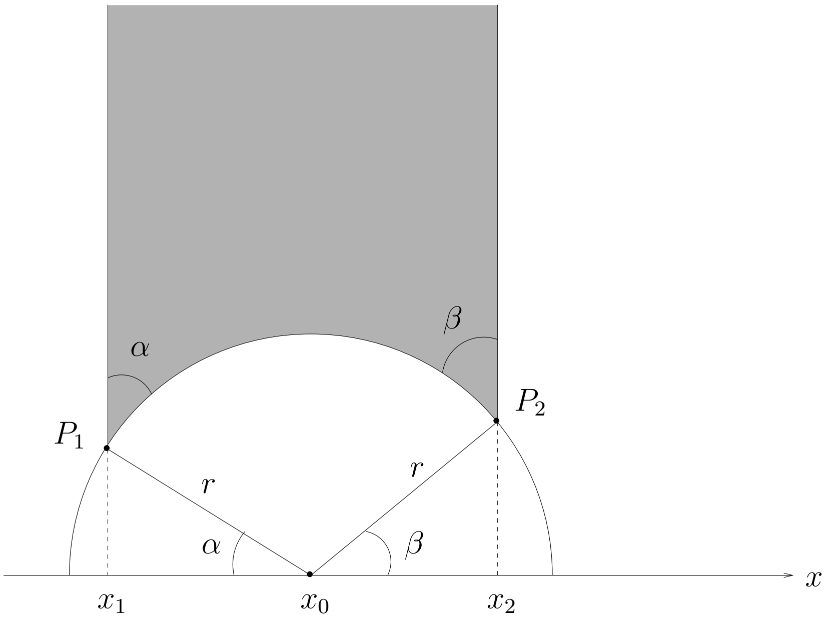
$$\int_{T^2} K \, d\mathrm{vol} = 0.$$

Does this come as a surprise to you or do you have an explanation?

Exercise 11. We derive the formula

$$\mathrm{vol}(\Delta P_1 P_2 P_3) = \pi - \alpha - \beta$$

for a triangle $\Delta P_1 P_2 P_3$ in the upper half space model $\mathbb{H}^2$ with an ideal vertex P_3 and interior angles α, β. Look at the following picture and denote the "ideal" point at infinity by P_3.



Show that

$$\mathrm{vol}(\Delta P_1P_2P_3) = \arcsin(\frac{x_0 - x_1}{r}) + \arcsin(\frac{x_2 - x_0}{r}).$$

(You may use $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.) Conclude from this with $\arcsin(x) + \arccos(x) = \pi/2$ that

$$\mathrm{vol}(\Delta P_1P_2P_3) = \pi - \alpha - \beta.$$