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Riemannian Geometry IV

Problems, set 7.

Exercise 14. Given a curve $c : [a,b] \to \mathbb{R}^3$, $c(t) = (f(t), 0, g(t))$ without self intersections and with $f(t) > 0$ for all $t \in [a,b]$, let $M \subset \mathbb{R}^3$ denote the surface of revolution obtained by rotating this curve around the vertical Z-axis. Let ∇ denote the Levi-Civita connection of M. An almost global coordinate chart is given by $\varphi : U \to V := (a,b) \times (0, 2\pi)$,

$$\varphi^{-1}(x_1, x_2) = (f(x_1)\cos x_2, f(x_1)\sin x_2, g(x_1)).$$

(a) Calculate the Christoffel symbols of this coordinate chart and express

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$

in terms of the basis $\frac{\partial}{\partial x_k}$.

(b) Let $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$. $\frac{D}{dt}$ denotes covariant derivative along γ_1. Calculate

$$\frac{D}{dt}\gamma_1'.$$

Show that this vector field along γ_1 vanishes if and only if the generating curve c of M is parametrized proportional to arc-length. Note that γ_1 is obtained by rotation of c by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized propotional to arc length.

(c) Let $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$. $\frac{D}{dt}$ denotes covariant derivative along γ_2. Calculate

$$\frac{D}{dt}\gamma_2'.$$

Show that this vector field along γ_2 vanishes if and only if $f'(x_1) = 0$. Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.

(d) Assume that the generating curve c of the surface of revolution is arc-length parametrized. Show in this particular case that

$$\mathrm{vol}(M) = 2\pi \int_a^b f(x_1)\, dx_1.$$