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Riemannian Geometry IV

Problems, set 8 (to be handed in on 22 January 2009).

Exercise 15. An almost global coordinate chart of the sphere $S^2 \subset \mathbb{R}^3$ is given by $\varphi : U \to V = (-\pi/2, \pi/2) \times (0, 2\pi)$, $U \subset S^2$,

$$\varphi^{-1}(x_1, x_2) = (\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1).$$

We want to determine the parallel vector fields along certain parallels of the sphere.

(a) Determine the Christoffel symbols with respect to this cordinate system. **Hint:** You may use the Solution of Exercise 14.

(b) Let X be the parallel vector field along $c_1 : (-\pi, \pi) \to S^2$, $c_1(t) = \varphi^{-1}(0, \pi + t)$ with $X(0) = \frac{\partial}{\partial x_1}|_{c_1(0)}$. Calculate X explicitely in terms of the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$.

(c) For $a, b > 0$ let

$$A = \begin{pmatrix} 0 & -a \\ b & 0 \end{pmatrix}.$$

Show that

$$\mathrm{Exp}(tA) = \begin{pmatrix} \cos(\sqrt{ab}t) & -\sqrt{\frac{a}{b}} \sin(\sqrt{ab}t) \\ \sqrt{\frac{b}{a}} \sin(\sqrt{ab}t) & \cos(\sqrt{ab}t) \end{pmatrix}.$$

(d) Let Y be the parallel vector field along $c_2 : (-\pi, \pi) \to S^2$, $c_2(t) = \varphi^{-1}(\pi/4, \pi + t)$ with $Y(0) = \frac{\partial}{\partial x_1}|_{c_2(0)}$. Calculate Y explicitely in terms of the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$.

Exercise 16. Let (M, g) be a Riemannian manifold and $c : [a, b] \to M$ be a differentiable curve. Let $\frac{D}{dt}$ denote the corresponding covariant derivative along the curve c. Show the following: For any two parallel vector fields X, Y along c we have

$$\frac{d}{dt}\langle X, Y \rangle \equiv 0,$$

i.e., the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry.

Hint: I am happy if you try to prove this statement in the particular case that the vector fields X, Y along c have global extensions $\tilde{X}, \tilde{Y} : M \to TM$.

However, be aware that not every vector field along a curve may have a global extension; an extreme example is the case where $c : [a, b] \to M$ is a constant map $c(t) = p$ for all $t \in [a, b]$ and $X(t)$ is varying in T_pM. In this case, you need to write X, Y with respect to a basis of a coordinate system. Therefore, it is convenient to assume that $c([a, b])$ is contained in the domain of a coordinate chart. But this assumption is not a serious restriction, for otherwise one covers $c([a, b])$ with a finite sequence of covering coordinate charts and argues locally (see the solution sheet).

Recall that the webpage for Riemannian Geometry IV with the script of this term's lectures and all exercises and solutions is at

`http://www.maths.dur.ac.uk/~dma0np/riemgeom0809/riemgeom.html`

MERRY CHRISTMAS !!!