Riemannian Geometry IV: MATH 4171

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1 Differentiable manifolds

In Differential Geometry the objects of interest were mainly surfaces in ambient $\mathbb{R}^3$. While they may have topology (like the two-dimensional torus), their main property is that they look, locally, like an open set of $\mathbb{R}^2$. This was the reason why, even though the surface of the Earth is topologically a sphere, it was believed for a long time that the earth is flat. We will now generalise the concept of surfaces to higher dimensions and obtain k-dimensional manifolds. We will see that for the definition of these objects we do not need an ambient Euclidean space. This was the point of view which Bernhard Riemann took in his inaugural lecture on 10 June, 1854, in Göttingen entitled

"On the hypotheses which lie at the foundations of geometry"

But this abstraction comes with a price: It will be harder to introduce tangent vectors and tangent spaces, which were just vectors or planes in Euclidean 3-space in the case of surfaces in $\mathbb{R}^3$. But it is still possible to define tangent spaces and geodesics and even curvature.

This abstraction is also very useful when trying to understand our universe, which in General Relativity is considered as a four-dimensional semi-Riemannian manifold with indefinite metric (the tangent space splits into three space dimensions and one time dimension).

1.1 Abstract manifolds

Let us start with some examples which will be seen later to be abstract differentiable manifolds:

Example 1 (surface of revolution)**.** *Let $\gamma : [a,b] \to \mathbb{R}^3$ be an injective differentiable curve with $\gamma(t) = (\gamma_1(t), 0, \gamma_3(t))$, $\gamma'(t) \neq 0$ and $\gamma_3(t) > 0$ for all $t \in (a,b)$. The associated surface of revolution is given by rotating this curve about the x_1-axis, i.e.,*

$$\begin{aligned} M &= \{(\gamma_1(t), \gamma_3(t)\cos\alpha, \gamma_3(t)\sin\alpha \mid t \in (a,b), \alpha \in [0,2\pi]\} \\ &= \{(x_1,x_2,x_3) \mid \exists t \in (a,b) \text{ such that } x_1 = \gamma_1(t), x_2^2 + x_3^2 = \gamma_3^2(t)\}. \end{aligned}$$

E.g., $\gamma(t) = (t,0,c)$, $c > 0$ gives a horizontal cylinder.

Example 2 (matrix groups)**.** *The matrix groups $SL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det A = 1\}$, $O(n) = \{A \in M(n,\mathbb{R}) \mid A \cdot A^\top = \mathrm{Id}\}$ and $SO(n) = \{A \in O(n) \mid \det A = 1\}$. They can be considered as submanifolds of the ambient space $\mathbb{R}^{n^2} \cong M(n,\mathbb{R})$.*

Example 3 (real projective space)**.** *The set of all lines through the origin in $\mathbb{R}^{n+1}$ is called the* real projective n-space $\mathbb{R}P^n$*. It can be identified with the set $S^n/\sim$, where $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $p \sim q$ iff $p = \pm q$: every line $L \subset \mathbb{R}P^n$ is in one-one correspondence with $S^n \cap L = \{p, -p\} \in S^n/\sim$.*

As mentioned before, our point of view is to disregard any ambient Euclidean space and to define differentiable manifolds intrinsically. We start our definition of an abstract manifold with the definition of an atlas:

Definition 1.1. *Let M be a set. An* atlas *of M is a collection of pairs $(U_\alpha, \varphi_\alpha)$ (α running through some finite/countable/uncountable index set $\mathcal{A}$) such that*

(a) $\bigcup_{\alpha\in\mathcal{A}} U_\alpha = m$, i.e. $(U_\alpha)_{\alpha\in\mathcal{A}}$ is a cover of M

(b) each φ_α is a bijective map $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$, where V_α is an open subset of $\mathbb{R}^n$, and, for every $\alpha, \beta \in \mathcal{A}$, $\varphi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ is also open. The maps φ_α are called coordinate charts *of M.*

(c) For any two indices $\alpha, \beta \in \mathcal{A}$ the coordinate change

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a (infinitely often) differentiable *map between open sets in $\mathbb{R}^n$.*

The integer n is called the dimension *of the space M.*

Remark. *Since $\varphi_\alpha \circ \varphi_\beta^{-1}$ is the inverse of $\varphi_\beta \circ \varphi_\alpha^{-1}$, the coordinate changes are bijective maps which are differentiable in both directions. Such maps are called* diffeomorphisms*.*

Next, we define open sets in a space M with an atlas and, thus, introduce a *topology* on M:

Definition 1.2. *Let M be a n-dimensional space with an atlas $(U_\alpha, \varphi_\alpha)$. A subset X of M is called* open*, if for every $\alpha \in \mathcal{A}$ the image $\varphi_\alpha(X \cap U_\alpha) \subset V_\alpha$ is an open subset of $\mathbb{R}^n$. An open subset $U \subset M$ containing a point $x \in M$ is called a* neighbourhood *of x.*

Note that $\emptyset$ and M are trivially open subsets of M. It can be shown that arbitrary unions and finite intersections of open sets of M are, again, open set of M.

To have an abstract differential manifold M, we finally need that the topological space M with an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ has the Hausdorff property:

Definition 1.3. *A set M is called a* (differentiable) manifold*, if is comes with a countable atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ and satisfies the*

Hausdorff property: *for $x, y \in M$, $x \neq y$, there are neighbourhoods $A_x, A_y \subset M$ of x and y, respectively, satisfying $A_x \cap A_y = \emptyset$ (i.e., different points in M can be separated by open sets).*

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Example 4 (a non-manifold)**.** *Let M be*

$$M = (\mathbb{R} \backslash \{0\}) \cup \{0_1, 0_2\}$$

with the coordinate charts

$$\begin{aligned} \varphi_j \quad &: \quad U_j := (\mathbb{R}\backslash\{0\}) \cap \{0_j\} \to \mathbb{R} \\ \varphi_j(x) \quad &= \quad \begin{cases} x, & \text{if } x \neq 0_j, \\ 0, & \text{if } x = 0_j. \end{cases} \end{aligned}$$

Then (U_1, φ_1) and (U_2, φ_2) are an atlas of M, but M is not Hausdorff since neighbourhoods of 0_1 and 0_2 contain the sets $A_{\epsilon,0_1} := \varphi_1^{-1}((-\epsilon, \epsilon))$ and $A_{\delta,)_2} := \varphi_2^{-1}((-\delta, \delta))$ for suitably small $\epsilon, \delta > 0$. But these sets $A_{\epsilon,0_1}$ and $A_{\delta.0_2}$ always intersect. Hence the Hausdorff property is violated and M is not a manifold.

We finish this section by looking at another example: the cartesian product of to manifolds:

Example 5 (cartesian product)**.** *Let M be a manifold of dimension m and N be a manifold of dimension n. Then the cartesian product*

$$M \times N := \{(x, y) \mid x \in M, y \in N\}$$

is a manifold of dimension $m + n$: If $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ and $(\tilde{U}_\alpha, \tilde{\varphi}_\beta)_{\beta \in \tilde{\mathcal{A}}}$ are atlases of M and N, respectively, then an atlas of $M \times N$ is given by $(U_\alpha \times \tilde{U}_\beta, \psi_{\alpha,\beta})_{(\alpha,\beta) \in \mathcal{A} \times \tilde{\mathcal{A}}}$, defined by

$$\begin{aligned} \psi_{\alpha,\beta} \quad &: \quad U_\alpha \times \tilde{U}_\beta \to V_\alpha \times \tilde{V}_\beta \subset \mathbb{R}^{m+n}, \\ \psi_{\alpha,\beta}(x, y) \quad &:= \quad (\varphi_\alpha(x), \tilde{\varphi}_\beta(y)). \end{aligned}$$

The coordinate changes are

$$\psi_{\gamma,\delta}^{-1} \circ \psi_{\alpha,\beta}(x, y) = (\varphi_\gamma^{-1} \circ \varphi_\alpha(x), \tilde{\varphi}_\delta^{-1} \circ \tilde{\varphi}_\beta(y)),$$

and obviously differentiable. Finally, we have to check the Hausdorff property: Let $(x,y) \neq (z,w)$. This means that $x \neq z$ or $y \neq w$. Choose open neighbourhoods U_x, U_z of $x, z \in M$ which don't intersect if $x \neq z$. Choose open neighbourhoods $\tilde{U}_y, \tilde{U}_w$ of $y, w \in N$ which don't intersect if $y \neq w$. One checks that $U_x \times \tilde{U}_y \subset M \times N$ and $U_z \times \tilde{U}_w \subset M \times N$ are open neighbourhoods of (x,y) and (z,w) and

$$(U_x \times \tilde{U}_y) \cap (U_z \times \tilde{U}_w) = \emptyset.$$

If we choose, e.g., $M = N = S^1$, $M \times N$ is topologically the 2-dimensional torus.

1.2 Manifolds and regular values

Many examples of differentiable manifolds are obtained as level set of a regular value:

Definition 1.4. *Let $f : U \to \mathbb{R}^k$, $U \subset \mathbb{R}^n$ open, be a differentiable map and*

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1}(x) & \ldots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}^k$$

be its Jacobi matrix at $x \in U$. $x \in U$ is called regular point *of f if $Df(x) : \mathbb{R}^n \to \mathbb{R}^k$ is surjective (i.e., if $Df(x)$ has rank k). $y \in \mathbb{R}^k$ is called a* regular value *of f, if every $x \in f^{-1}(y) \subset U$ is a regular point.*

Theorem 1.5. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$, $k \leq n$, be differentiable. Let $y \in \mathbb{R}^k$ be a regular value of f. Then the preimage $f^{-1}(y) \subset U \subset \mathbb{R}^n$ is a differentiable manifold of dimension $n - k$.*

Example 6 (sphere)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|^2 = \sum_{i=1}^n x_i^2$. Then $1 \in \mathbb{R}$ is a* regular value *of f:*

$$Df(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \ldots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix} = \begin{pmatrix} 2x_1 & \ldots & 2x_n \end{pmatrix} : \mathbb{R}^n \to \mathbb{R}.$$

Here, $Df(x)$ is surjective iff $Df(x) \neq 0$. But $Df(x) = 2x \neq 0$ iff $x \neq 0$. Hence, if $f(x) = 1 = \|x\|^2$, then $Df(x) \neq 0$. Thus

$$f^{-1}(1) = \{x \in \mathbb{R}^n \mid \|x\| = 1\} = S^{n-1}$$

is a $(n-1)$-dimensional manifold.

The proof of Theorem 1.5 is a direct application of the *Implicit Function Theorem*, which we recall for convenience: Let $f : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ be differentiable and $f(x^1, x^2) = y$. Then the Jacobi matrix $Df(x^1, x^2)$ consists of two components

$$Df(x^1, x^2) = \left(\frac{\partial f}{\partial x^1}(x^1, x^2) \quad \frac{\partial f}{\partial x^2}(x^1, x^2) \right),$$

where $\frac{\partial f}{\partial x^2}(x^1, x^2) \in M(k, \mathbb{R})$. Assume that $\det \frac{\partial f}{\partial x^2}(x^1, x^2) \neq 0$ (which would mean, if x^1, x^2 would be just one-dimensional, that the graph of f over the point (x^1, x^2) would be strictly increasing or decreasing in the x^2-direction). Then the level set $f^{-1}(y)$ can be parametrised, at least locally, by the first coordinate x^1, i.e., there is a differentiable function $\psi : V_1 \to V_2$, defined on open neighbourhoods $V_1 \subset \mathbb{R}^{n-k}$ and $V_2 \subset \mathbb{R}^k$ of x^1 and x^2, respectively, such that

$$f^{-1}(y) \cap (V_1 \times V_2) = \{(x, \psi(x)) \mid x \in V_1\}.$$

Sketch of Proof of Theorem 1.5. We explain how to construct the coordinate charts of $M = f^{-1}(y) \subset \mathbb{R}^n$. Let $x \in M$. Since $Df(x) : \mathbb{R}^n \to \mathbb{R}^k$ has rank k, there exists an invertible $k \times k$-submatrix. Permuting the coordinates (and thus also the columns of $Df(x)$), we can assum that

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_{n-k}}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_{n-k}}(x) & \cdots & \frac{\partial f_k}{\partial x_n}(x) \end{pmatrix} : \mathbb{R}^k \to \mathbb{R}^k$$

has non-zero determinant. Let $x = (x^1, x^2) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. By the implicit function theorem there exists a differentiable function $\psi : V \to \mathbb{R}^k$, defined on an open neighbourhood $V \subset \mathbb{R}n - k$ of x^1, such that $M = f^{-1}(y)$, in a neighbourhood of x, if the graph of ψ over V, i.e., can be parametrised by the injective map

$$\phi : V \to M \subset \mathbb{R}^n, \quad \phi(z) = (z, \psi(z)) \in \mathbb{R}^{n-k} \times \mathbb{R}^k.$$

Let $U = \phi(V) \subset M$. The inverse map $\varphi := \phi^{-1} : U \to V \subset \mathbb{R}^{n-k}$, is then a coordinate chart of the manifold M. It remains to prove differentiability of coordinate changes, which we omit. $\square$

Before we consider the next example, we recall the following fact: Let $f : \mathbb{R}^n \to \mathbb{R}^k$. Then $Df(x) : \mathbb{R}^n \to \mathbb{R}^k$ and we have

$$Df(x)(z) = \lim_{t \to 0} \frac{f(x + tz) - f(x)}{t} \in \mathbb{R}^k. \tag{1}$$

This follows easily via the chain rule, when we introduce the curve $c(t) = x + tz$ and consider the j-component of $DF(x)(z)$, i.e.,

$$(Df(x)(z))_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) z_i = (f_j \circ c)'(0) = \lim_{t \to 0} \frac{f_j(c(t)) - f_j(x)}{t}.$$

Example 7 ($SO(n)$ as a manifold)**.** *We show that* $SO(n) = \{A \in M(n, \mathbb{R}) \mid AcdotA^\top = \mathrm{Id}, \det A = 1\}$ *is a manifold. The set* $GL^+(n) := \{A \in M(n, \mathbb{R}) \mid \det A > 0\}$ *is an open subset of* $M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$*, since* $\det : M(n, \mathbb{R}) \to \mathbb{R}$ *is continuous. Note that* $A \cdot A^\top = \mathrm{Id}$ *implies* $\det A = \pm 1$*, so*

$$SO(n) = O(n) \cap GL^+(n).$$

We consider the map

$$\begin{aligned} f \quad &: \quad GL^+(n) \to Sym(n) = \{C \in M(n, \mathbb{R}) \mid C^\top = C\} \cong \mathbb{R}^{\frac{n(n+1)}{2}}, \\ f(A) \quad &= \quad A \cdot A^\top - \mathrm{Id}. \end{aligned}$$

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Then $SO(n) = f^{-1}(0)$ *and we only have to show that* $0 \in Sym(n)$ *is a regular value of* f. *Let* $A \in f^{-1}(0) = SO(n)$. *We have to show that the linear map*

$$Df(A) : M(n, \mathbb{R}) \to Sym(n)$$

is surjective. *From* (1) *we conclude that*

$$\begin{aligned} Df(A)(B) &= \lim_{t\to 0} \frac{f(A+tB) - f(A)}{t} = \lim_{t\to 0} \left((A+tB)\cdot(A+tB)^\top - A \cdot A^\top\right) \\ &= \lim_{t\to 0} \frac{1}{t}\left(t(AB^\top + BA^\top) + t^2 BB^\top\right) = AB^\top + BA^\top. \end{aligned}$$

Let $C \in Sym(n)$, *i.e.,* $C^\top = C$. *Then, using* $AA^\top = \mathrm{Id}$,

$$\begin{aligned} Df(A)(\frac{1}{2}CA) &= A(\frac{1}{2}CA)^\top + (\frac{1}{2}CA)A^\top \\ &= \frac{1}{2}(AA^\top)c^\top + \frac{1}{2}C(AA^\top) = \frac{1}{2}C^\top + \frac{1}{2}C = C. \end{aligned}$$

Hence, $Df(A)$ *is surjective for all* $A \in f^{-1}(0)$ *and* $SO(n)$ *is an* $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$*-dimensional manifold.*

1.3 Differentiable maps, tangent vectors and differentials

Next we consider maps between manifolds. They are called differentiable if they are differentiable with respect to all coordinate charts:

Definition 1.6. *Let* M *and* N *be a* m*-dimensional, respectively,* n*-dimensional manifold. A map* $f : M \to N$ *is called* differentiable at $x \in M$, *if there exist coordinate charts* $\varphi : U \to V \subset \mathbb{R}^m$, $x \in U \subset M$, $\tilde{\varphi} : \tilde{U} \to \tilde{V} \subset \mathbb{R}^n$, $f(x) \in \tilde{U} \subset N$, *such that*

$$\tilde{\varphi} \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(\tilde{u})) \to \tilde{V}$$

is (infinitely many times) differentiable at $\varphi(x) \in V$. *(Note that this definition does not depend on the choices of the coordinate charts, since coordinate changes are differentiable.) A function* $f : M \to N$ *is called* differentiable*, if* f *is differentiable at all points* $x \in M$. *The set of all differentiable functions is denoted by* $C^\infty(M)$ *and carries in a natural way the structure of a* $\mathbb{R}$*-vector space.*

Remark. *Since* $\mathbb{R}^n$ *is trivially a manifold (with one global coordinate chart, the identity), differentiability of maps* $\mathbb{R}^n \to M$ *or* $M \to \mathbb{R}^n$ *is included in the above definition. Differentiable maps* $c : (a, b) \to M$ *are called* curves *in* M.

Our next goal is to define the derivative of a map $f : M \to N$, the differential Df. The differential will be a *linear map between tangent spaces*. The tangent space T_xM of a surface $M \subset \mathbb{R}^3$ at $x \in M$ can be easily visualised: First choose the unique affine plane V tangent to M at x. Then T_xM is the plane parallel to V and passing through the origin. This makes T_xM a two-dimensional linear subspace of $\mathbb{R}^3$. The lack of an ambient Euclidean space $\mathbb{R}^N$ in the case of an abstract differentiable manifold makes it more difficult to introduce tangent vectors and tangent spaces. We explain how to overcome this problem by first introducing *directional derivatives* along curves:

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Definition 1.7. *Let M be a manifold and $x \in M$. Let*

$$D(M, x) = \{f : M \to \mathbb{R} \mid f \text{ is differentiable at } x\}.$$

Let $c : (a, b) \to M$ be a curve with $c(t_0) = x$. The directional derivative *of f along c at $x = c(t_0)$ is denoted by $c'(t_0)(f) \in \mathbb{R}$ and defined as*

$$c'(t_0)(f) := \lim_{t\to 0} \frac{f(c(t_0 + t)) - f(x)}{t} = (f \circ c)'(t_0) = \frac{d}{dt}|_{t=t_0} f \circ c.$$

Note that $D(M, x)$ is an algebra over $\mathbb{R}$, i.e., a vector space over $\mathbb{R}$, since for $\lambda \in \mathbb{R}$, $f, g \in D(M, x)$ we have $\lambda f, f + g \in D(M, x)$ and we also have $f \cdot g \in D(M, x)$. The directional derivative along c at $x = c(t_0)$ has the following properties, which can be easily checked:

(a) $c'(t_0)(\lambda f + g) = \lambda c'(t_0)(f) + c'(t_0)(g)$.

(b) $c'(t_0)(fg) = f(x)c'(t_0)(g) + g(x)c'(t_0)(f)$ (product rule).

A map $D(M, x) \to \mathbb{R}$ with the properties (a) and (b) is called a *linear derivation* of the algebra $D(M, x)$.

Example 8 (directional derivatives)**.** *Note that different curves $c_1, c_2 : \mathbb{R} \to M$ with $c_1(0) = c_2(0) = x$ can define the same directional derivative. For example: $c_1, c_2 : \mathbb{R} \to \mathbb{R}^2$, $c_1(t) = (t, 0)$ and $c_2(t) = (t, t^2)$, we have*

$$\begin{aligned} c_1'(0)(f) &= (f \circ c_1)'(0) = \frac{\partial f}{\partial x_1}(0), \\ c_2'(0)(f) &= \frac{d}{dt}|_{t=0} f(t, t^2) = \frac{\partial f}{\partial x_1}(0) \cdot 1 + \frac{\partial f}{\partial x_2}(0) \cdot 2t|_{t=0} = \frac{\partial f}{\partial x_1}(0). \end{aligned}$$

So the directional derivative $c'(0)$ depends only on the direction of $c(t)$ at $t = 0$ and not on its global behaviour. Thinking of $c_1'(0)$ and $c_2'(0)$ as elements in $\mathbb{R}^2$, we obtain

$$c_1'(0) = (1, 0) = c_2'(0).$$

One can generally check: If $c_1, c_2 : (a, b) \to M$ and $x = c_1(t_0) = c - 2(t_0)$, then

$$c_1'(t_0) = c_2'(t_0) \text{ as linear derivations}$$

if and only if there is a coordinate chart $\varphi : U \to V \subset \mathbb{R}^n$, $x \in U \subset M$ such that

$$(\varphi \circ c_1)'(t_0) = (\varphi \circ c_2)'(t_0) \text{ as ordinary vectors in } \mathbb{R}^n.$$

Moreover, if this holds for one coordinate chart, then it holds for all coordinate charts (U, φ) with $x \in U$.

Conclusion and Outlook: Every curve through $\in M$ defines a directional derivative $c'(t_0) : D(M, x) \to \mathbb{R}$. Different curves can have the same directional derivative. The set of directional derivatives will form the *tangent space* T_xM. At the moment it is not at all clear whether T_xM is a linear vector space, i.e., whether we can add directional derivatives. We certainly cannot simply add curves!

Definition 1.8. *Let M be a manifold and $x \in M$. A* tangent vector *of M at x is the directional derivative $c'(t_0) : D(M, x) \to \mathbb{R}$ of a curve $c : (a, b) \to M$ with $c(t_0) = x$. The set of all tangent vectors at x defines the tangent space $T_x M$ of M at the point x.*

Next we introduce particular tangent vectors associated to the coordinate directions of a chart:

Let M be a n-dimensional manifold and $\varphi : U \to V \subset \mathbb{R}^n$, $U \subset M$ a coordinate chart. Let $\varphi = (x_1, \dots, x_n)$, i.e., $x_i : U \to \mathbb{R}$ is the i-th coordinate function of φ. The *i-th coordinate curve* through $p \in U$ is given by

$$c_i : (-\epsilon, \epsilon) \to U \subset M, \quad c_i(t) = \varphi^{-1}(\varphi(p) + te_i),$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. Note that $c_i(0) = p$ Let $\frac{\partial}{\partial x_i}|_p := c_i'(0) \in T_p M$ be the corresponding tangent vector, i.e.,

$$\frac{\partial}{\partial x_i}|_p(f) = \frac{d}{dt}|_{t=0}(f \circ c_i)(t) = \frac{d}{dt}|_{t=0}(f \circ \varphi^{-1})(\varphi(p) + te_i) = \frac{\partial(f \circ \varphi)}{\partial x_i}(\varphi(p)),$$

where the right most expression is ordinary i-th partial derivative of the function $f \circ \varphi : (-\epsilon, \epsilon) \to \mathbb{R}^n$ at the point $\varphi(p)$.

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The tangent vectors in coordinate directions are important to prove that $T_p M$ is a n-dimensional vector space.

Proposition 1.9. *Let M be a n-dimensional manifold. Then $T_p M$ carries the structure of a n-dimensional vector space.*

Proof. We show that $\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p$ form a basis of $T_p M$.

(a) Let $c : (a, b) \to M$ be a curve with $c(0) = p$. We show that $c'(0) : D(M, p) \to \mathbb{R}$ is a linear combination of the tangent vectors $\frac{\partial}{\partial x_i}|_p$ of a coordinate chart $\varphi = (x_1, \dots, x_n) : U \to V \subset \mathbb{R}^n$, $\in U \subset M$: Let $\varphi \circ c =: (c_1, \dots, c_n) : (a, b) \to V \subset \mathbb{R}^n$. Then, applying the chain rule, we obtain

$$c'(0)(f) = (f \circ c)'(0) = ((f \circ \varphi^{-1}) \circ (\varphi \circ c))'(0) =$$
$$\sum_{i=1}^n \frac{\partial(f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot c_i'(0) = \sum_{i=1}^n c_i'(0) \cdot \frac{\partial}{\partial x_i}|_p(f).$$

(b) Any linear combination $\sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}|_p$ is a tangent vector: Let

$$c : (-\epsilon, \epsilon) \to M, \qquad c(t) = \varphi^{-1}(\varphi(p) + t(\alpha_1, \dots, \alpha_n)).$$

Then $c(0) = p$ and

$$(c_1(t), \dots, c_n(t)) = \varphi \circ c(t) = \varphi(p) + t(\alpha_1, \dots, \alpha_n),$$

which implies that $c_i'(0) = \alpha_i$ and

$$c'(0)(f) = \sum_{i=1}^n c_i'(0) \frac{\partial}{\partial x_i}|_p(f) = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}|_p(f).$$

(c) $\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p$ are linear independent: Let

$$\sum \alpha_i \frac{\partial}{\partial x_i}|_p = O : D(M,p) \to \{0\}.$$

Choose $\phi \in C^\infty(M)$ with $\phi \equiv 1$ near $p \in M$ and $\phi \equiv 0$ outside $U \subset M$. Then $f_j := \phi \cdot x_j$ is well defined on all of M (not only on the domain $U \subset M$ of φ), namely

$$f_j(q) = \begin{cases} \phi(q)x_j(q), & \text{if } q \in U, \\ 0, & \text{if } q \notin U, \end{cases}$$

and $f_j \in C^\infty(M)$. Morevoer, $f_j \circ \varphi^{-1}(x_1, \dots, x_n) = x_j$ near $\varphi(p)$ and, consequently,

$$\frac{\partial}{\partial x_i}|_p = \frac{f_j \circ \varphi^{-1}}{\partial x_i}(\varphi(p)) = \delta_{ij}.$$

Thus, $\sum \alpha_i \frac{\partial}{\partial x_i}|_p = 0$ implies $\sum \alpha_i \frac{\partial}{\partial x_i}|_p(f_j) = \sum \alpha_i \delta_{ij} = \alpha_j = 0$ for all j and $\frac{\partial}{\partial x_1}|_p, \dots, \frac{\partial}{\partial x_n}|_p$ are linear independent.

□

Remark. *If a differentiable manifold M^n is contained in an Euclidean space $\mathbb{R}^N$ (i.e., is a submanifold of $\mathbb{R}^N$), then we can identify the abstract tangent vectors $c'(0) \in T_pM$, $c'(0) : D(M,p) \to \mathbb{R}$ with the classical tangent vectors $c'(0) \in \mathbb{R}^N$ via*

$$\mathbb{R}^N \ni c'(0) = \begin{pmatrix} c'(0)(y_1) & \dots & c'(0)(y_N) \end{pmatrix},$$

where $c'(0)$ on the left side in the above formula is just the classical tangent vector in $\mathbb{R}^N$ and the occurences of $c'(0)$ on the right side are presenting the map $c'(0) : D(M,p) \to \mathbb{R}$, applied to the functions y_j, which are the restrictions of the j-th coordinate functions $\mathbb{R}^N \to \mathbb{R}$, $(a_1, \dots, a_n) \mapsto a_j$, to the domain $M \subset \mathbb{R}^N$.

Before we consider our next example, we need the following useful lemma:

Lemma 1.10. *Let $A : (-\epsilon, \epsilon) \to GL(n, \mathbb{R})$ be a curve. Then*

$$(\det A)'(t) = (\det A(t)) \cdot \mathrm{tr}(A^{-1}(t)A'(t)).$$

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Proof. Let $A(t) = \begin{pmatrix} a_1(t) & \dots & a_n(t) \end{pmatrix}$ and $a_i(t) = \begin{pmatrix} a_{1i}(t) & \dots & a_{ni}(t) \end{pmatrix}^\top$ be the columns of $A(t)$. Then

$$\det A = \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n},$$

where the a_{ij} are functions $(\epsilon, \epsilon) \to \mathbb{R}$. Using the product rule, we obtain

$$\begin{aligned} (\det A)' &= \sum_{\sigma \in \mathcal{S}_n} \mathrm{sgn}(\sigma) \left(a'_{\sigma(1)1} \cdots a_{\sigma(n)n} + \dots + a_{\sigma(1)1} \cdots a'_{\sigma(n)n} \right) \\ &= \det \begin{pmatrix} a'_1 & \dots & a_n \end{pmatrix} + \dots + \det \begin{pmatrix} a_1 & \dots & a'_n \end{pmatrix}. \end{aligned}$$

Since $\det A(t) \neq 0$, $a_1(t), \dots, a_n(t)$ are a basis of $\mathbb{R}^n$. Therefore, there exist functions $\alpha_{ij}(t)$ such that

$$a'_j(t) = \sum_{i=1}^{n} \alpha_{ij}(t) a_i(t), \tag{2}$$

i.e. (using the fact that the determinant doesn't change when adding multiples of one column to another column of the matrix),

$$\begin{aligned}
(\det A)' &= \det\big((\alpha_{11}a_1) \quad \dots \quad a_n\big) + \dots + \det\big(a_1 \quad \dots \quad (\alpha_{nn}a_n)\big) \\
&= \left(\sum_{i=1}^{n} \alpha_{ii}\right) \cdot \det A = \det A \cdot \mathrm{tr}(\alpha_{ij}).
\end{aligned}$$

Note that (2) is equivalent to $A'(t) = A(t) \cdot (\alpha_{ij}(t))$, i.e., $(\alpha_{ij}(t)) = A^{-1}(t)A'(t)$, and thus

$$(\det A)'(t) = (\det A(t)) \cdot \mathrm{tr}(A^{-1}(t)A'(t)).$$

□

Example 9 (tangent space of $SL(n,\mathbb{R})$)**.** *Let $I \in SL(n,\mathbb{R})$ denote the identity matrix. We know from Exercise 2 that $SL(n,\mathbb{R})$ is a differentiable manifold of dimension n^2-1. Therefore, $T_I SL(n,\mathbb{R}) \subset M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ is a subspace of dimension n^2-1. Let*

$$A : (-\epsilon, \epsilon) \to SL(n,\mathbb{R})$$

be a curve with $A(0) = I$. So $\det A(t) = 1$ and Lemma 1.10 implies that

$$0 = (\det A)'(0) = \underbrace{(\det A(0))}_{=1} \cdot \mathrm{tr}(\underbrace{A^{-1}(0)}_{=I} A'(0)) = \mathrm{tr}(\underbrace{A'(0)}_{\in T_I SL(n,\mathbb{R})}).$$

We conclude that $T_I SL(n,\mathbb{R}) \subset \{B \in M(n,\mathbb{R}) \mid \mathrm{tr}B = 0\}$. Since $\mathrm{tr}B = 0$ means the linear condition $\sum_i b_{ii} = 0$, we conclude that $\{B \in M(n,\mathbb{R}) \mid \mathrm{tr}B = 0\}$ is of dimension n^2-1. By dimension arguments we therefore have

$$T_I SL(n,\mathbb{R}) = \{B \in M(n,\mathbb{R}) \mid \mathrm{tr}B = 0\}.$$

Definition 1.11. *A differentiable manifold G is called a* Lie group*, if G is also a group and if the group operations*

$$\begin{aligned}
G \times G &\to G, \qquad (g_1, g_2) \mapsto g_1 g_2, \\
G &\to G, \qquad g \mapsto g^{-1}
\end{aligned}$$

are differentiable.

Examples of Lie groups are $GL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det A \neq 0\}$ (dimension n^2), $SL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det A = 1\}$ (dimension n^2-1), $O(n) = \{A \in M(n,\mathbb{R}) \mid AA^\top = I\}$ (dimension $\frac{n(n-1)}{2}$), $SO(n) = \{A \in O(n) \mid \det A = 1$ (dimension $\frac{n(n-1)}{2}$). We also know that

$$\begin{aligned}
T_I SL(n,\mathbb{R}) &= \{B \in M(n,\mathbb{R}) \mid \mathrm{tr}B = 0\}, \\
T_I SO(n) &= \{B \in M(n\mathbb{R}) \mid B^\top = -B\} \text{ (see Exercise 4).}
\end{aligned}$$

Now, we are able to define the derivative of a differentiable function $f : M \to N$:

Definition 1.12. *Let M, N be differentiable manifolds and $f : M \to N$ be a differentiable map. For $p \in M$, the map*

$$Df(p) : T_pM \to T_{f(p)}N, \quad c'(t) \mapsto (f \circ c)'(t),$$

is called the differential of f at p *and is a linear map.*

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Remark. *Let $v \in T_pM$ be a tangent vector, i.e., a linear derivation $v : D(M,p) \to \mathbb{R}$. Then we have $Df(p)(v) : D(N, f(p)) \to \mathbb{R}$ and*

$$Df(p)(v)(g) = v(g \circ f) \qquad \forall g \in D(N, f(p)). \tag{3}$$

Proof of (3)*: Let $c : (-\epsilon, \epsilon) \to M$ be a curve with $c(0) = p$ and $c'(0) = v$. Then*

$$\begin{aligned} Df(p)(c'(0))(g) &= (f \circ c)'(0)(g) = \frac{d}{dt}|_{t=0}(g \circ (f \circ c)(t) \\ &= \frac{d}{dt}|_{t=0}((g \circ f) \circ c)(t) = c'(t)(g \circ f). \end{aligned}$$

Proof of linearity of the differential Df(p). Using the statement of the previous remark and the fact, that T_pM is a vector space, we have

$$\begin{aligned} (Df)(p)(\alpha v_1 + v_2)(g) = (\alpha v_1 + v_2)(g \circ f) &= \alpha v_1(g \circ f) + v_2(g \circ f) \\ &= \alpha Df(p)(v_1)(g) + Df(p)(v_2)(g), \end{aligned}$$

i.e.,

$$Df(p)(\alpha v_1 + v_2) = \alpha Df(p)(v_1) + Df(p)(v_2).$$

□

Example 10 (differential)**.** *Let*

$$\begin{aligned} S^2 &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}, \\ Z &= \{(x, y, z) \mid x^2 + y^2 = 1, -1 < z < 1\}, \end{aligned}$$

and $f : Z \to S^2$, $f(x, y, z) = (\sqrt{1 - z^2}x, \sqrt{1 - z^2}y, z)$. Let $p = (1, 0, z_0) \in Z$ and $v_1 = (0, 1, 0), v_2 = (0, 0, 1) \in T_pZ$. Curves $c_1, c_2 : (-\epsilon, \epsilon) \to Z$, representing v_1 and v_2 are

$$\begin{aligned} c_1(t) &= (\cos t, \sin t, z_0) \in Z, \\ c_2(t) &= (1, 0, z_0 + t) \in Z, \end{aligned}$$

since $c_1(0) = p = c_2(p)$ and $c_1'(0) = v_1$, $c_2'(0) = v_2$. Then $f(p) = (\sqrt{1 - z_0^2}, 0, z_0)$. We calculate $Df(p)(v_1), Df(p)(v_2)$. The image curves are

$$\begin{aligned} (f \circ c_1)(t) &= (\sqrt{1 - z_0^2} \cos t, \sqrt{1 - z_0^2} \sin t, z_0), \\ (f \circ c_2)(t) &= (\sqrt{1 - (z_0 + t)^2}, 0, z_0 + t). \end{aligned}$$

Differentiating the image curves gives

$$\begin{aligned} (f \circ c_1)'(t) &= (-\sqrt{1 - z_0^2} \sin t, \sqrt{1 - z_0^2} \cos t, 0), \\ (f \circ c_2)'(t) &= (\frac{-(z_0 + t)}{\sqrt{1 - (z_0 + t)^2}}, 0, 1), \end{aligned}$$

so

$$Df(p)(v_1) = (f \circ c_1)'(0) = (0, \sqrt{1 - z_0^2}, 0) \in T_{f(p)}S^2 = (f(p))^\perp$$

and

$$Df(p)(v_2) = (f \circ c_2)'(0) = (\frac{-z_0}{\sqrt{1 - z_0^2}}, 0, 1) \in T_{f(p)}S^2 = (f(p))^\perp.$$

1.4 Vector fields and Lie brackets

The union of all tangent spaces of a differentiable manifold form the tangent bundle. The tangent bundle is, again, a differentiable manifold:

Definition 1.13. *Let M be a n-dimensional differentiable manifold. The tangent spaces T_pM, $p \in M$, are all pairwise disjoint (since they consist of linear maps on different spaces $D(M,p)$) and their disjoint union*

$$TM := \bigcup_{p \in M} T_x M$$

is called the tangent bundle *of M. There is a canonical* footpoint projection *$\pi : TM \to M$, $\pi(v) = p$ if $v \in T_pM$.*

Proposition 1.14. *The tangent bundle TM of a n-dimensional manifold M carries the structure of a $(2n)$-dimensional manifold and the footpoint projection $\pi : TM \to M$ is a differentiable map.*

Proof. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathcal{A}}$ be an atlas of M. Let $\varphi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha) : U_\alpha \to V_\alpha \subset \mathbb{R}^n$, i.e., $x_1^\alpha, \ldots, x_n^\alpha$ are the coordinate functions of φ_α. We construct an atlas of TM by choosing, for every $\alpha \in \mathcal{A}$, the subsets $\tilde{U}_\alpha \subset TM$:

$$\tilde{U}_\alpha := \pi^{-1}(U_\alpha) = \bigcup_{p \in U_\alpha} T_pM,$$

and the bijective maps

$$\begin{aligned} \psi_\alpha \quad &: \quad \tilde{U}_\alpha \to V_\alpha \times \mathbb{R}^n \subset \mathbb{R}^{2n} \\ \psi_\alpha\left(\sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i^\alpha}|_p\right) \quad &:= \quad (\varphi_\alpha(p), \beta_1, \ldots, \beta_n). \end{aligned}$$

The inverse map ψ_α^{-1} is then given by

$$\psi_\alpha^{-1}(x, \beta) = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i^\alpha}|_{\varphi^{-1}(x)}.$$

Obviously, we have $\bigcup_{\alpha \in \mathcal{A}} \tilde{U}_\alpha = TM$ and for the coordinate changes with

$\varphi_\alpha(p) = x$ we obtain, by using Exercise 3,

$$\begin{aligned}
\psi_\gamma \circ \psi_\alpha^{-1}(x,\beta) &= \psi_\gamma \left(\sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i^\alpha}|_p \right) \\
&= \psi_\gamma \left(\sum_{i=1}^n \beta_i \left(\sum_{j=1}^n \frac{\partial (x_j^\gamma \circ \varphi_\alpha^{-1})}{\partial x_i}(x) \cdot \frac{\partial}{\partial x_j^\gamma}|_p \right) \right) \\
&= \psi_\gamma \left(\sum_{j=1}^n \left(\sum_{i=1}^n \beta_i \frac{\partial (x_j^\gamma \circ \varphi_\alpha^{-1})}{\partial x_i}(x) \right) \frac{\partial}{\partial x_j^\gamma}|_p \right) \\
&= \left(\varphi_\gamma \circ \varphi_\alpha^{-1}(x), \beta \cdot \left(\frac{\partial (x_j^\gamma \circ \varphi_\alpha^{-1})}{\partial x_i}(x) \right)_{1 \le i,j \le n} \right),
\end{aligned}$$

which is obviously differentiable. Now we look at the projection $\pi : TM \to M$. For every $p \in M$, there exists a $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$, and the composition $\psi_\alpha^{-1} \circ \pi \circ \varphi_\alpha : V_\alpha \times \mathbb{R}^n \to V_\alpha$ is simply

$$\psi_\alpha^{-1} \circ \pi \circ \varphi_\alpha(x,\beta) = x,$$

which is differentiable. Hence, π is differentiable. □

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Next, we introduce vector fields.

Definition 1.15. *A vector field X on a differentiable manifold M is a map $M \to TM$, which associates to every $p \in M$ a tangent vector $X(p) \in T_P M$. We also require that $X; M \to TM$ is a differentiable map, i.e., the tangent vectors $X(p)$ vary smoothly with varying footpoints $p \in M$. The space of all vector fields on a manifold M is denoted by $\mathcal{X}(M)$.*

Remarks. *(a) Note for every vector field $X : M \to TM$ on a differentiable manifold M that the composition $\pi \circ X : M \to M$ is the identity.*

(b) If X is a vector field on M and (U, φ) a coordinate chart with $\varphi = (x_1, \dots, x_n)$, then the vector field X assumes, locally, the form

$$X(p) = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x_i}|_p \in T_p M$$

with differentiable functions $f_i : U \to \mathbb{R}$.

(c) The space $\mathcal{X}(M)$ of all vector fields on a manifold M carries the structure of an infinite-dimensional vector space, since for $\lambda \in \mathbb{R}$ and $X, Y \in \mathcal{X}(M)$ we can define addition by

$$(X+Y)(p) := X(p) + Y(p) \in T_p M$$

and scalar multiplication by

$$(\lambda X)(p) := \lambda X(p).$$

Example 11 (vector field on S^2). *A vector field on* $S^2 = \{u \in \mathbb{R}^3 \mid \|u\| = 1\}$ *is given by*

$$X(u) := (2u_3 - u_2, u_1, -2u_1) \in T_u S^2.$$

Note that $T_u S^2 = u^\perp$*, so* $X(u) \in T_u S^2$ *holds true if and only if* $\langle X(u), u\rangle = 0$*:*

$$\langle X(u), u\rangle = \langle (2u_3 - u_2, u_1, -2u_1), (u_1, u_2, u_3)\rangle = u_1(2u_3 - u_2) + u_2 u_1 - 2u_3 u_1 = 0.$$

A coordinate chart (U, φ) *of* S^2 *is given by*

$$\varphi^{-1} : (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi) \ni (x_1, x_2) \mapsto (\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1).$$

We like to express X *in terms of* $\frac{\partial}{\partial x_i}$*: Let*

$$p = \varphi^{-1}(x_1, x_2) = (\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1).$$

We have

$$\begin{aligned} \frac{\partial}{\partial x_1}|_p &= (-\sin x_1 \cos x_2, -\sin x_1 \sin x_2, \cos x_1), \\ \frac{\partial}{\partial x_2}|_p &= (-\cos x_1 \sin x_2, -\cos x_1 \cos x_2, 0) \end{aligned}$$

and

$$X(p) = (2 \sin x_1 - \cos x_1 \cos x_2, \cos x_1 \cos x_2, -2 \cos x_1 \cos x_2).$$

Comparing the last coordinate, we see that

$$X = -2 \cos x_2 \frac{\partial}{\partial x_1} + \lambda \frac{\partial}{\partial x_2}.$$

This implies that

$$\begin{aligned} \lambda \frac{\partial}{\partial x_2} &= X + 2 \cos x_2 \frac{\partial}{\partial x_1} \\ &= (2 \sin x_1 \sin^2 x_2 - \cos x_1 \sin x_2, \cos x_2(\cos x_1 - 2 \sin x_1 \sin x_2), 0) \\ &= \frac{\partial}{\partial x_2} + 2(\sin x_1 \sin^2 x_2, -\sin x_1 \sin x_2 \cos x_2, 0) \\ &= (1 - 2 \tan x_1 \sin x_2) \frac{\partial}{\partial x_2}. \end{aligned}$$

Thus we conclude

$$X = -2 \cos x_2 \frac{\partial}{\partial x_1} + (1 - 2 \tan x_1 \sin x_2) \frac{\partial}{\partial x_2}.$$

Remark. *A vector field* $X : M \to TM$ *determines, at each point* $p \in M$*, a directional derivative* $X(p) : D(M, p) \to \mathbb{R}$*. So, for a function* $f \in C^\infty(M)$*, we obtain a new function* $Xf : M \to \mathbb{R}$*, defined by*

$$(Xf)(p) := \underbrace{X(p)}_{\in T_p M}(\underbrace{f}_{\in C^\infty(M) \subset D(M,p)}) \in \mathbb{R}.$$

We check that Xf is also differentiable: if $X = \sum g_i \frac{\partial}{\partial x_i}$ is a local description of X with respect to a coordinate chart $\varphi = (x_1, \ldots, x_n)$, then

$$Xf = \sum g_i \frac{\partial f}{\partial x_i}$$

locally, where $\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(p))$. We then have

$$(Xf) \circ \varphi^{-1} = \sum g_i \circ \varphi^{-1} \frac{\partial (f \circ \varphi^{-1})}{\partial x_i},$$

which is differentiable in the classical sense.

Next we define the Lie bracket of two vector fields:

Proposition 1.16. *Let X, Y be two vector fields on a differentiable manifold M. Then there exists a unique vector field $Z \in \mathcal{X}(M)$, such that, for all $f \in C^\infty(M)$,*

$$Zf = X(Yf) - Y(Xf).$$

We denote this vector field as the Lie bracket *$[X, Y]$.*

Proof. We calculate $X(Yf) - Y(Xf)$ in local coordinates: Let $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum b_i \frac{\partial}{\partial x_i}$. Then

$$\begin{aligned}
X(Yf) &= X\left(\sum_i b_i \frac{\partial f}{\partial x_i}\right) = \sum_j a_j \frac{\partial}{\partial x_j}\left(\sum_i b_i \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{i,j} a_j \left(\frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + b_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right) \\
&= \sum_{i,j} a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum_{i,j} a_j b_i \frac{\partial^2 f}{\partial x_j \partial x_i}.
\end{aligned}$$

Similarly,

$$Y(Xf) = \sum_{i,j} b_i \frac{\partial a_i}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum_{i,j} a_j b_i \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Note that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(p) = \frac{\partial^2 (f \circ \varphi^{-1}}{\partial x_j \partial x_i}(\varphi(p)) = \frac{\partial^2 (f \circ \varphi^{-1}}{\partial x_i \partial x_j}(\varphi(p)) = \frac{\partial^2 f}{\partial x_i \partial x_j}(p),$$

since classical partial derivatives commute. So

$$\begin{aligned}
X(Yf) - Y(Xf) &= \sum_{i,j}\left(a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\frac{\partial f}{\partial x_j}\right) \\
&= \sum_i \left(\sum_j a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial f}{\partial x_i},
\end{aligned}$$

and Z is locally given as the vector field

$$Z = [X, Y] = \sum_i \left(\sum_j a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\frac{\partial}{\partial x_i}.$$

□

Proposition 1.17 (properties of the Lie bracket)**.** *Let $X, Y, Z \in \mathcal{X}(M)$ be vector fields on a differentiable manifold M. Then*

(a) $[X,Y] = -[Y,X]$,

(b) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ *for all* $a, b \in \mathbb{R}$,

(c) Jacobi's identity:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0,$$

(d) for all functions $f, g \in C^\infty(M)$*:*

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

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Proof. Properties (a)-(c) are Exercise 6. For d), note that the product rule (see also the text following the Definition 1.7) yields

$$Z(fg) = g \cdot (Zf) + f \cdot (Zg).$$

Using this fact, we conclude that

$$\begin{aligned}
[fX, gY]h &= fX(g(Yh)) - gY(f(Xh)) \\
&= f(Xg)(Yh) + fgX(Yh) - g(Yf)(Xh) - gfY(Xh) \\
&= fg(X(Yh) - Y(Xh)) + f(Xg)(Yh) - g(Yf)(Xh) \\
&= fg[X,Y]h + f(Xg)Yh - g(Yf)Xh,
\end{aligned}$$

which implies that

$$[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X.$$

□

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2 Basics about Riemannian manifolds

So far, we introduced differentiable manifolds, their tangent spaces and vectors, the tangent bundle, vector fields and Lie brackets. In order to **start doing geometry**, we need to have measurements for the lengths of tangent vectors and angles between them. This will be provided via the concept of a **Riemannian metric**. Once, we have equipped a manifold with a Riemannian metric, we can define the volume of subsets of a manifold, the lengths of curves on the manifold and define the distance between points of the manifold.

2.1 Riemannian metrics

Definition 2.1. *Let M be a differentiable manifold. A* Riemannian metric *$g = \{g_p\}_{p \in M}$ is a family of symmetric inner products*

$$g_p : T_pM \times T_pM \to \mathbb{R},$$

which depend smoothly on $p \in M$, i.e., for any two vector fields $X, Y \in \mathcal{X}(M)$ the function $p \mapsto g_p(X(p), Y(p))$ is differentiable. We often use the notation

$$\langle v, w \rangle_p = g_p(v, w) \qquad v, w \in T_pM$$

for the Riemannian metric. A differentiable manifold M together with a Riemannian metric g is called a Riemannian manifold *(M, g).*

Example 12 (Euclidean space $\mathbb{R}^n$ as Riemannian manifold)**.** *$M = \mathbb{R}^n$ is a differentiable manifold with one global coordinate chart, the identity. The tangent spaces $T_pM = T_p\mathbb{R}^n$ can be canonically identified with pairwise disjoint copies of R^n via*

$$\begin{aligned} c : (-\epsilon, \epsilon) &\to \mathbb{R}^n, \qquad c(0) = p, \\ c'(0) \in T_p\mathbb{R}^n &\rightsquigarrow c'(0) = \begin{pmatrix} c_1'(0) & \dots & c_n'(0) \end{pmatrix}^\top \in \mathbb{R}^n. \end{aligned}$$

(If one would like to stress the pairwise disjointness of the different $T_p\mathbb{R}^n$, one could identify $T_p\mathbb{R}^n$ with $\{p\} \times \mathbb{R}^n$.) A Riemannian metric $g_p : T_p \times T_pM \to \mathbb{R}$ is given by the standard inner product

$$g_p(v, w) = \langle v, w \rangle = \sum_{i=1}^n v_i w_i.$$

Example 13 (surface in $\mathbb{R}^3$ as Riemannian manifold)**.** *Let $M \subset \mathbb{R}^3$ be a surface. The tangent spaces T_pM can be canonically identified with 2-dimensional subspaces of $\mathbb{R}^3$. T_pM inherits a natural inner product from $\mathbb{R}^3$, the so-called* first fundamental form. *This inner product defines a* Riemannian metric *on M. Let $M = S^2$, $p = (0, 0, 1)^\top \in S^2$,*

$$v = (v_1, v_2, 0),\ w = (w_1, w_2, 0) \in T_pS^2.$$

Then $g_p(v, w) = v_1 w_1 + v_2 w_2$.

Definition 2.2. *Let (M, g) be a Riemannian manifold. The* length *of a tangent vector $v \in T_pM$ is defined by*

$$\|v\|_p := \sqrt{g_p(v, v)}.$$

Let (M, g) be a Riemannian manifold and $\varphi : U \to V \subset \mathbb{R}^n$, $U \subset M$ be a coordinate chart with coordinate functions $x_1, \dots, x_n : U \to \mathbb{R}$. In such a situation, we will often look at the differentiable functions $g_{ij} : U \to \mathbb{R}$, $1 \le i, j \le n$, defined by

$$g_{ij}(p) := \langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p \rangle_p.$$

In the special case of a parametrised surface $M \subset \mathbb{R}^3$ the functions g_{ij} of the associated coordinate chart coincide with the functions E, F, G of the first fundamental form:

$$g_{11} = E, \qquad g_{12} = g_{21} = F, \qquad g_{22} = G.$$

Let us now introduce a very important Riemannian manifold, the n-dimensional *hyperbolic space*, and discuss three different models of this space: the *hyperboloid model*, the *Poincaré ball model* and the *upper half space model*. These models seems to descibe completely different Riemannian manifolds, but we will see bit later that all three models are isometric and represent therefore the same space.

Example 14 (n-dimensional hyperbolic space)**.** *There are different models of the this Riemannian manifold:*

(a) Hyperboloid model: *Consider the following indefinite symmetric form q on $\mathbb{R}^{n+1}$:*

$$q(y,z) := \left(\sum_{i=1}^{n} y_i z_i\right) - y_{n+1} z_{n+1}$$

and define

$$\mathbb{W}^n := \{y \in \mathbb{R}^{n+1} \mid q(y,y) = -1, y_{n+1} > 0\}.$$

We can think of $\mathbb{W}^n$ as a n-dimensional submanifold of $\mathbb{R}^{n+1}$ and identify $T_p\mathbb{W}^n$ with a n-dimensional subspace of $\mathbb{R}^{n+1}$. We define for $v, w \in T_p\mathbb{W}^n \subset \mathbb{R}^{n+1}$:

$$g_p(v,w) = q(v,w).$$

It turns out that this restriction of q to $T_p\mathbb{W}^n$ is always positive definite on $T_p\mathbb{W}^n$, i.e., defines a Riemannian metric on $\mathbb{W}^n$.

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Case $n = 2$**:** *An almost global coordinate chart is given by*

$$\begin{aligned} \varphi^{-1} &: (0,2\pi) \times (0,\infty) \to \mathbb{W}^n, \\ \varphi^1(x_1,x_2) &= (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2), \end{aligned}$$

since the image of φ^{-1} covers almost all of $\mathbb{W}^2$, except for the curve obtained by the intersection of $\mathbb{W}^2 \subset \mathbb{R}^3$ with the half plane $x_1 \geq 0$, $x_2 = 0$. Let $p = \varphi^{-1}(x_1,x_2)$. Then

$$\begin{aligned} \frac{\partial}{\partial x_1}|_p &= \begin{pmatrix} -\sin x_1 \sinh x_2 & \cos x_1 \sinh x_2 & 0 \end{pmatrix}^\top, \\ \frac{\partial}{\partial x_2}|_p &= \begin{pmatrix} \cos x_1 \cosh x_2 & \sin x_1 \cosh x_2 & \sinh x_2 \end{pmatrix}^\top. \end{aligned}$$

A straightforward calculation yields

$$(g_{ij}(p)) = \begin{pmatrix} g_p\left(\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_1}|_p\right) & g_p\left(\frac{\partial}{\partial x_1}|_p, \frac{\partial}{\partial x_2}|_p\right) \\ g_p\left(\frac{\partial}{\partial x_2}|_p, \frac{\partial}{\partial x_1}|_p\right) & g_p\left(\frac{\partial}{\partial x_2}|_p, \frac{\partial}{\partial x_2}|_p\right) \end{pmatrix} = \begin{pmatrix} \sinh^2 x_2 & 0 \\ 0 & 1 \end{pmatrix},$$

which is a positive definite matrix.

(b) Poincaré ball model: *Let*

$$\mathbb{B}^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\},$$

where $\|x\|^2 = \sum_i x_i^2$*. Since* $\mathbb{B}^n$ *is an open subset of* $\mathbb{R}^n$*, the tangent spaces* $T_x\mathbb{B}^n$ *can be canonically identified with* $\mathbb{R}^n$*. We introduce the Riemannian metric*

$$g_x(v_1, v_2) := \frac{4}{(1 - \|x\|^2)^2} \langle v_1, v_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ *denotes the Euclidean inner product of* $\mathbb{R}^n$*.*

(c) Upper half space model: *Let*

$$\mathbb{H}^n := \{x \in \mathbb{R}^n \mid x_n > 0\}.$$

The tangent spaces $T_x\mathbb{H}^n$ *can again be canonically identified with* $\mathbb{R}^n$*. We introduce the Riemannian metric*

$$g_x(v_1, v_2) := \frac{\langle v_1, v_2 \rangle}{x_n^2},$$

where $\langle \cdot, \cdot \rangle$ *denotes the Euclidean inner product of* $\mathbb{R}^n$*.*

Before we introduce the notion of an isometry, let us first recall the notion of a *linear isometry* between two inner product vector spaces $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$: An isomorphism $T : V_1 \to V_2$ is called a *linear isometry* if we have

$$\langle Tv, Tw \rangle_2 = \langle v, w \rangle \quad \text{for all } v, w \in V_1.$$

Definition 2.3. *Let* (M_1, g_1) *and* (M_2, g_2) *be two Riemannian manifolds. A bijective differentiable map* $f : M_1 \to M_2$ *with differentiable inverse* $f^{-1} : M_2 \to M_1$ *is called a* diffeomorphism*. (This forces the manifolds* M_1, M_2 *to have the same dimension.) A diffeomorphism* $f : M_1 \to M_2$ *is called an* isometry*, if for all* $p \in M_1$ *the differential*

$$Df(p) : T_pM_1 \to T_{f(p)}M_2$$

is a linear isometry, i.e.,

$$\langle Df(p)v, Df(p)w \rangle_{2,f(p)} = \langle v, w \rangle_{1,p} \quad \textit{for all } v, w \in T_pM. \tag{4}$$

Remark. *It is sufficient to check* (4) *for* $v = w$*, since*

$$\langle v, w \rangle_p = \frac{1}{4}(\langle v + w, v + w \rangle_p - \langle v - w, v - w \rangle_p) = \frac{1}{4}(\|v + w\|_p^2 - \|v - w\|_p^2).$$

Example 15 (isometries between models of the hyperbolic space)**.** *(a) The isometry* $f : \mathbb{W}^n \to \mathbb{B}^n$ *is as follows: Let* $\mathbb{B}^n$ *be represented as the unit disc with center* $0 \in \mathbb{R}^{n+1}$ *in the hyperplane* $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$*. Then every Euclidean straight ray starting from* $(0, -1) \in \mathbb{R}^n \times \mathbb{R}$ *through a point in* $\mathbb{W}^n$ *intersects* $\mathbb{B}^n$ *in a unique point (and vice versa). The so-defined map* $f : \mathbb{W}^n \to \mathbb{B}^n$ *can be seen to be an isometry between these two spaces (see Exercise 7).*

(b) In the case $n = 2$*, we have* $\mathbb{B}^2, \mathbb{H}^2 \subset \mathbb{R}^2 \cong \mathbb{C}$*. We show that*

$$f : \mathbb{H}^2 \to \mathbb{B}^2, \quad f(z) = \frac{z - i}{z + i}$$

is an isometry. It is easy to see that f *is a diffeomorphism with inverse*

$$f^{-1}(z) = \frac{z + 1}{iz - i}.$$

Let $z = x + iy \in \mathbb{H}^2$ *and* $v = v_1 + iv_2 \in T_z\mathbb{H}^2 \cong \mathbb{C}$. *Then*

$$g_z^{\mathbb{H}^2}(v, v) = \frac{\langle v, v\rangle}{y^2} = \frac{v_1^2 + v_2^2}{y^2}.$$

v *is represented by the curve* $c : (-\epsilon, \epsilon) \to \mathbb{H}^2$, $c(t) = z + tv$, *i.e.,* $v = c'(0)$. *We have*

$$\begin{aligned} Df(z)(v) &= (f \circ c)'(0) = \frac{d}{dt}|_{t=0} \frac{c(t) - i}{c(t) + i} \\ &= \frac{d}{dt}|_{t=0} \frac{(z - i) + tv}{(z + i) + tv} = \frac{v((z + i) + tv) - v((z - i) + tv)}{((z + i) + tv)^2}|_{t=0} \\ &= \frac{2iv}{(z + i)^2} = \frac{2i}{(z + i)^2} v \in T_{f(z)}\mathbb{B}^2. \end{aligned}$$

Introducing $w = Df(z)(v)$, *for simplicity, we obtain*

$$\begin{aligned} g_{f(z)}^{\mathbb{B}^2}(w, w) &= \frac{4}{(1 - |f(z)|^2)^2} \langle \frac{2i}{(z + i)^2} v, \frac{2i}{(z + i)^2} v\rangle \\ &= \frac{16}{|z + i|^4 (1 - \frac{|z-i|^2}{|z+i|^2})^2} \langle v, v\rangle \\ &= \frac{16}{(|z + i|^2 - |z - i|^2)^2} \langle v, v\rangle = \frac{16}{(x^2 + (y + 1)^2 - x^2 - (y - 1)^2)^2} \langle v, v\rangle \\ &= \frac{16}{(4y)^2} \langle v, v\rangle = \frac{v_1^2 + v_2^2}{y^2} \\ &= g_z^{\mathbb{H}^2}(v, v), \end{aligned}$$

which shows that $Df : T_z\mathbb{H}^2 \to T_{f(z)}\mathbb{B}^2$ *is a linear isometry.*

Lemma 2.4. *Let* (M_1, g_1) *and* (M_2, g_2) *be two Riemannian manifolds and* $f : M_1 \to M_2$ *be a diffeomorphism. Let* $\varphi : M_1 \to V \subset \mathbb{R}^n$ *be a global coordinate chart with coordinate functions* $x_1, \dots, x_n$, *i.e.,* $\varphi = (x_1, \dots, x_n)$. *Then* $\psi := \varphi \circ f^{-1} : M_2 \to V$ *is a global coordinate chart of* M_2. *Let* $\psi = (y_1, \dots, y_n)$, *i.e.,* $y_i = x_i \circ f^{-1}$. *Then* f *is an isometry if we have*

$$\langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\rangle_{1,p} = \langle \frac{\partial}{\partial y_i}|_{f(p)}, \frac{\partial}{\partial y_j}|_{f(p)}\rangle_{2,f(p)} \qquad \text{for all } p \in M_1. \tag{5}$$

Proof. (a) We first prove $Df(p)(\frac{\partial}{\partial x_i}|_p) = \frac{\partial}{\partial y_i}|_{f(p)}$: Let $h \in D(M_2, f(p))$. Then

$$\begin{aligned} Df(p)(\frac{\partial}{\partial x_i}|_p)(h) &= \frac{\partial}{\partial x_i}|_p (h \circ f) \\ &= \frac{\partial (h \circ f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \\ &= \frac{\partial (h \circ \psi^{-1})}{\partial x_i}(\psi(f(p))) \\ &= \frac{\partial}{\partial y_i}|_{f(p)}(h). \end{aligned}$$

(b) Now we assume that (5) holds. Let $v = \sum a_i \frac{\partial}{\partial x_i}|_p \in T_pM_1$. Then, using (a), we obtain

$$\begin{aligned}
\langle v, v\rangle_{1,p} &= \sum_{i,j} a_i a_j \langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\rangle_{1,p} = \sum_{i,j} a_i a_j \langle \frac{\partial}{\partial y_i}|_{f(p)}, \frac{\partial}{\partial y_j}|_{f(p)}\rangle_{2,f(p)} \\
&= \langle \sum a_i \frac{\partial}{\partial y_i}|_{f(p)}, \sum a_j \frac{\partial}{\partial y_j}|_{f(p)}\rangle_{2,f(p)} \\
&= \langle Df(p)(\sum a_i \frac{\partial}{\partial x_i}|_p), Df(p)(\sum a_j \frac{\partial}{\partial x_j}|_p)\rangle_{2,f(p)} \\
&= \langle Df(p)(v), Df(p)(v)\rangle_{2,f(p)},
\end{aligned}$$

showing that $Df(p) : T_pM_1 \to T_{f(p)}M_2$ is a linear isometry. $\square$

2.2 Integration on Riemannian manifolds

Let (M, g) be a Riemannian manifold and φ a coordinate chart with coordinate functions $x_1, \dots, x_n$. Recall that the functions g_{ij} are defined as

$$g_{ij} : U \to \mathbb{R}, \qquad g_{ij}(p) = \langle \frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\rangle_p.$$

Definition 2.5. *Let $f : M \to \mathbb{R}$ be a functions with $\{p \in M \mid f(p) \neq 0\} \subset U$ and $\varphi : U \to V$ a coordinate chart as above. Then we define*

$$\int_M f \,\mathrm{dvol} := \int_U f \,\mathrm{dvol} := \int_V f \circ \varphi^{-1}(x) \sqrt{\det(g_{ij} \circ \varphi^{-1}(x))}\, dx. \qquad (6)$$

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The set $\{p \in M \mid f(p) \neq 0\}$ of a function $f : M \to \mathbb{R}$ is called the *support* $\operatorname{supp} f$ of the function f (in most references, the support is actually defined to be the closure of this set). Definition 2.5 introduces the integral of a function with support in a coordinate chart. Later on, we will explain how to define the integral of general functions.

We have to check whether the definition (6) is *independent of the choice of coordinate chart*. Let $\psi : U \to V' \subset \mathbb{R}^n$ be a second coordinate chart with coordinate functions $y_1, \dots, y_n$. Then $F := \psi \circ \varphi^{-1} : V \to V'$ is a diffeomorphism and the *transformation rule* states that

$$\int_{V'} g(y)\, dy = \int_V g(F(x)) |\det(DF(x))|\, dx.$$

Let $g'_{ij}(p) := \langle \frac{\partial}{\partial y_i}|_p, \frac{\partial}{\partial y_j}|_p\rangle_p$. We know from Exercise 3 that

$$\frac{\partial}{\partial x_i}|_p = \sum_k \frac{\partial y_k}{\partial x_i}(p) \frac{\partial}{\partial y_k}|_p.$$

This implies that

$$\begin{aligned}
g_{ij}(p) &= \sum_{k,l} \frac{\partial y_k}{\partial x_i}(p) \frac{\partial y_l}{\partial x_j}(p) \langle \frac{\partial}{\partial y_k}|_p, \frac{\partial}{\partial y_l}|_p\rangle \\
&= \sum_{k,l} \frac{\partial y_k}{\partial x_i}(p) \frac{\partial y_l}{\partial x_j}(p) g'_{kl}(p).
\end{aligned}$$

In matrix notation, we hence obtain

$$(g_{ij}(p)) = \left(\frac{\partial y_i}{\partial x_j}(p)\right)^\top (g'_{ij}(p)) \left(\frac{\partial y_i}{\partial x_j}(p)\right).$$

On the other hand, we have with $x = \varphi(p)$

$$DF(x) = \begin{pmatrix} \frac{\partial y_1 \circ \varphi^{-1}}{\partial x_1}(x) & \dots & \frac{\partial y_1 \circ \varphi^{-1}}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial y_n \circ \varphi^{-1}}{\partial x_1}(x) & \dots & \frac{\partial y_n \circ \varphi^{-1}}{\partial x_n}(x) \end{pmatrix}.$$

Thus we obtain

$$\frac{\partial y_i}{\partial x_j}(p) = \frac{\partial y_i \circ \varphi^{-1}}{\partial x_j}(x) = (DF)_{i,j}(x),$$

and the above matrix identity translates into

$$(g_{ij}(p)) = DF(x)^\top (g'_{ij}(p)) DF(x),$$

and

$$\sqrt{\det(g_{ij}(p))} = \sqrt{\det(g'_{ij}(p))} \cdot |\det DF(x)|.$$

Applying the transformation formula to (6) yields

$$\begin{aligned} \int_V f \circ \varphi^{-1} \sqrt{\det(g_{ij} \circ \varphi^{-1})}\, dx &= \int_V \underbrace{f \circ \varphi^{-1} \sqrt{\det(g'_{ij} \circ \varphi^{-1})}}_{g \circ \psi^{-1} \circ F} |\det DF| dx \\ &= \int_{V'} g \circ \psi^{-1}(y)\, dy = \int_{V'} f \circ \psi^{-1} \sqrt{\det(g'_{ij} \circ \psi^{-1})}\, dy, \end{aligned}$$

where we used the notation $g = f\sqrt{\det(g'_{ij})}$. This shows invariance of the expression under coordinate changes.

Definition 2.6. *Let (M, g) be a Riemannian manifold with a global coordinate chart $\varphi = (x_1, \dots x_n) : M \to \mathbb{R}^n$. The* volume *of a subset $A \subset M$ is defined as*

$$\mathrm{vol}(A) := \int_M \mathbf{1}_A \,\mathrm{dvol} = \int_A \mathrm{dvol} = \int_{\varphi(A)} \sqrt{\det(g_{ij} \circ \varphi^{-1}(x))}\, dx,$$

where $\mathbf{1}_A$ denotes the characteristic function of A, i.e.,

$$\mathbf{1}_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0. & \text{if } x \in M \backslash A. \end{cases}$$

Example 16 (volume calculation in the hyperbolic plane)**.** *Recall that $\mathbb{H}^2 = \{z \mid \mathrm{Im}(z) > 0\}$ with*

$$\langle v, w \rangle_z = \frac{v_1 w_1 + v_2 w_2}{y^2}.$$

A global coordinate chart is given by $\varphi : \mathbb{H}^2 \to \mathbb{R}^2$, $\varphi(x+iy) = (x,y)$ *and the functions* g_{ij} *are*

$$\begin{aligned} g_{11}(z) &= \langle \frac{\partial}{\partial x}|_z, \frac{\partial}{\partial x}|_z \rangle_z = \langle e_1, e_1 \rangle_z = \frac{1}{y^2}, \\ g_{12}(z) &= \langle \frac{\partial}{\partial x}|_z, \frac{\partial}{\partial y}|_z \rangle_z = \langle e_1, e_2 \rangle_z = 0 = g_{21}(z), \\ g_{22}(z) &= \langle \frac{\partial}{\partial y}|_z, \frac{\partial}{\partial y}|_z \rangle_z = \langle e_2, e_2 \rangle_z = \frac{1}{y^2}. \end{aligned}$$

Let $A = \{z = x+iy \in \mathbb{H}^2 \mid 0 < a \le y \le b, -n \le x \le n\}$. *The volume* $\mathrm{vol}(A)$ *is then*

$$\begin{aligned} \mathrm{vol}(A) &= \int_{-n}^{n} \int_a^b \sqrt{\det(g_{ij}(x+iy))}\, dy\, dx \\ &= \int_{-n}^{n} \int_a^b \left(\det \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix} \right)^{1/2} dy\, dx = \int_{-n}^{n} \int_a^b \frac{dy}{y^2}\, dx \\ &= \int_{-n}^{n} \left(\frac{1}{a} - \frac{1}{b} \right) dx = 2n(\frac{1}{a} - \frac{1}{b}). \end{aligned}$$

This shows that the following "boxes" all have the same hyperbolic volume in $\mathbb{H}^2$*:*

$$\begin{aligned} A_1 &= \{z = x+iy \in \mathbb{H}^2 \mid 2 \le y \le 4, -4 \le x \le 4\}, \\ A_2 &= \{z = x+iy \in \mathbb{H}^2 \mid 1 \le y \le 2, -2 \le x \le 2\}, \\ A_3 &= \{z = x+iy \in \mathbb{H}^2 \mid 1/2 \le y \le 1, -1 \le x \le 1\}, \\ A_4 &= \{z = x+iy \in \mathbb{H}^2 \mid 1/4 \le y \le 1/2, -1/2 \le x \le 1/2\}, \\ &\vdots \end{aligned}$$

In order to define the integral of functions which are not supported inside the domain of a single coordinate chart, we introduce the *partition of unity*.

Definition 2.7. *Let* M *be a differentiable manifold. A* partition of unity *on* M *is a collection* $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}$ *of differentiable functions* $\psi_\alpha : M \to [0,1]$ *such that*

(a) for every $p \in M$ *there exists an open neighbourhood* $U_p \subset M$ *such that*

$$\psi_\alpha|_{U_p} \neq 0 \quad \text{for only finitely many } \alpha \in \mathcal{A},$$

(b) we have

$$\sum_{\alpha \in \mathcal{A}} \psi_\alpha(p) = 1$$

for all $p \in M$.

Property (a) guarantees that at each point $p \in M$*, the sum* $\sum_{\alpha \in \mathcal{A}} \psi_\alpha(p)$ *has only finitely many non-zero terms.*

Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ *be an open cover of* M*. We say a partition of unity* $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}$ *is* subordinated *to* $\{U_\alpha\}$*, if for all* $\alpha \in \mathcal{A}$*:*

$$\mathrm{supp}\, \psi_\alpha = \{x \in M \mid \psi_\alpha(x) \neq 0\} \subset U_\alpha.$$

We have the following fact, which we state without a proof: 21 November 2008

Theorem 2.8. *Let M be a differentiable manifold and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ a countable atlas. Then there exists a partition of unity which is subordinated to $\{U_\alpha\}$.*

This theorem allows us to define integrals of general functions $f : M \to \mathbb{R}$:

Definition 2.9. *Let M be a Riemannian manifold, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ a countable atlas and $\{\psi_\alpha\}_{\alpha \in \mathcal{A}}$ a subordinated partition of unity. For any function $f : M \to \mathbb{R}$, we define*

$$\int_M f \, \mathrm{dvol} = \sum_{\alpha \in \mathcal{A}} \int_M f \psi_\alpha \, \mathrm{dvol},$$

where $\operatorname{supp} f\psi_\alpha \subset U_\alpha$ *and we use Definition 2.5 for the expression* $\int_M f\psi_\alpha \, \mathrm{dvol}$.

Note that property (b) of the partition of unity guarantees that we have

$$f = \sum_\alpha f \psi_\alpha.$$

It can easily be seen that the above definition doesn't depend on the choice of partition of unity. In most concrete examples we don't need a partition of unity, since there will be an almost global coordinate chart missing out only a union of finitely many lower dimensional sets in M, which don't contribute to the integral at all.

2.3 Riemannian manifolds as metric spaces

We first define the *length* of a curve $c : [a, b] \to M$:

Definition 2.10. *Let (M, g) be a Riemannian manifold and $c : [a, b] \to M$ a differentiable curve. The* length $L(c)$ *of c is then defined as*

$$L(c) := \int_a^b \|c'(t)\|_{c(t)} \, dt,$$

where $\|v\|_p = \sqrt{\langle v, v\rangle_p}$. If $c : [a, b] \to M$ is piecewise diferentiable, *i.e., c is continuous and we have a partititon $a = t_0 < t_1 < \cdots < t_n = b$ such that the restrictions $c|_{(t_i, t_{i+1})}$ are differentiable, then*

$$L(c) := \int_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \|c'(t)\|_{c(t)} \, dt.$$

Theorem 2.11. *Let $c : [0, T] \to M$ be a differentiable curve and $\gamma : [0, S] \to M$ be a* reparametrisation *of c, i.e., there exists a strictly monotone differentiable function $\varphi : [0, S] \to [0, T]$ with $\varphi(0) = 0$, $\varphi(S) = T$, such that $\gamma(s) = c(\varphi(s))$. Then*

$$L(c) = L(\gamma),$$

i.e., the length is invariant under reparametrisation.

Proof. The chain rule yields $\gamma'(s) = \varphi'(s)c'(\varphi(s))$: We have for $f \in D(M, \gamma(s))$:

$$\begin{aligned}\gamma'(s)(f) &= (f \circ \gamma)'(s) = (f \circ c \circ \varphi)'(s) \\ &= (f \circ c)'(\varphi(s)) \cdot \varphi'(s) = \varphi'(s)c'(\varphi(s))(f)\end{aligned}$$

This implies

$$L(\gamma) = \int_0^s \|\gamma'(s)\|_{\gamma(s)}\, ds = \int_0^s \varphi'(s)\|c'(\varphi(s))\|_{c(\varphi(s))}\, ds$$
$$\int_0^T \|c'(t)\|_{c(t)}\, dt = L(c),$$

where we also used $\varphi' > 0$. ☐

The length also remains unchanged under reparametrisations $\varphi : [0, S] \to [0, T]$ with $\varphi(0) = T$ and $\varphi(S) = 0$ and $\varphi' < 0$ (i.e., orientation reversing reparametrisations).

Definition 2.12. *A differentiable curve $c : [a, b] \to M$ is called* arc-length-parametrised *if*

$$\|c'(t)\|_{c(t)} = 1 \qquad \textit{for all } t \in [a, b].$$

Lemma 2.13. *For an arc-length parametrised curve $c : [a, b] \to M$ we have*

$$L(c|_{[a,t]}) = t - a.$$

Proof. We have

$$L(c|_{[a,t]}) = \int_a^t \|c'(t)\|_{c(s)}, ds = t - a.$$

☐

Proposition 2.14. *Every differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ has an arc-length reparametrisation $\gamma : [0, L(c)] \to M$.*

Proof. Let

$$l(t) := \int_a^t \|c'(s)\|_{c(s)}\, ds.$$

Then $l'(t) = \|c'(t)\|_{c(t)} > 0$, so $l : [a, b] \to [0, L(c)]$ is a strictly monotone, differentiable, bijective function. We have $l^{-1} : [0, L(c)] \to [a, b]$ and

$$\begin{aligned} l(l^{-1}(s)) &= s, \\ l'(l^{-1}(s))(l^{-1})'(s) &= 1, \\ (l^{-1})'(s) &= \frac{1}{l'(l(^{-1}(s))} = \frac{1}{\|c'(l^{-1}(s))\|_{c(l^{-1}(s))}}. \end{aligned}$$

Then $\gamma = c \circ l^{-1} : [0, L(c)] \to M$ is a reparametrisation of c satisfying

$$\gamma'(s) = c'(l^{-1}(s)) \cdot (l^{-1})'(s),$$

and thus

$$\|\gamma'(s)\|_{\gamma(s)} = \|c'(l^{-1}(s))\|_{c(l^{-1}(s))} \cdot \frac{1}{\|c'(l^{-1}(s))\|_{c(l^{-1}(s))}} = 1.$$

This means that γ is an arc-length reparametrisation of c. ☐

Example 17 (length calculation in the hyperbolic plane). *Let* $c : [a, b] \to \mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, $c(t) = ti$. *Then*

$$L(c) = \int_a^b \|c'(t)\|_{c(t)}\, dt = \int_a^b \frac{|i|}{t}\, dt = \int_a^b \frac{dt}{t} = \ln b - \ln a = \ln \frac{b}{a}.$$

We show for any differentiable curve $\gamma : [0, T] \to \mathbb{H}^2$ *with* $\gamma(0) = ai$, $\gamma(T) = bi$:

$$L(\gamma) \geq \ln \frac{b}{a} = L(c),$$

i.e., c is a shortest curve connecting ai and bi: Let $\gamma(t) = x(t) + iy(t)$:

$$\begin{aligned} L(\gamma) &= \int_0^T \|\gamma'(t)\|_{\gamma(t)}\, dt = \int_0^T \frac{\sqrt{(x')^2(t) + (y')^2(t)}}{y(t)}\, dt \geq \\ &\geq \int_0^T \frac{\sqrt{(y')^2(t)}}{y(t)}\, dt = \int_0^T \frac{|y'(t)|}{y(t)}\, dt \geq \int_0^T \frac{y'(t)}{y(t)}\, dt = \\ &= \int_0^T (\ln y)'(t)\, dt = \ln y(T) - \ln y(0) = \ln \frac{b}{a}. \end{aligned}$$

The length of curves allows us to define a distance function on a Riemannian manifold, which introduces a *metric space* structure on (M, g):

Definition 2.15. *Let (M, g) be a connected Riemannian manifold, i.e., for $p, q \in M$ there exists a curve $c : [a, b] \to M$ with $c(a) = p$, $c(b) = q$. We define the* distance function $d_g : M \times M \to [0, \infty)$ *as follows:*

$$d_g(p, q) := \inf\{L(c) \mid c : [a, b] \to M \text{ piecewise differentiable with } c(a) = p,\ c(b) = q\}.$$

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Remark. *A distance function $d : X \times X \to [0, \infty)$ satisfies the following three properties:*

(a) $d(p, q) = 0$ is equivalent to $p = q$,

(b) $d(p, q) = d(q, p)$,

(c) triangle inequality: $d(p_1, p_3) \geq d(p_1, p_2) + d(p_2, p_3)$.

(property (a) is **not** *immediately obvious for d_g, whereas (b) and (c) are easily seen to hold for d_g). The pair (X, d) is called a* metric space.

Example 18 (non-existence of distance-realising curve). *For two points $p, q \in M$ we call a curve $c : [a, b] \to M$ with $c(a) = p$, $c(b) = q$ and $L(c) = d(p, q)$ a* distance-realising *curve. Note however that such a curve doesn't need to exist: If we choose*

$$M = \mathbb{R}^2 \setminus \{0\}$$

with the standard Euclidean metric. Then $d((-1, 0), (1, 0)) = 2$, but there is no curve $c : [a, b] \to M$ connecting $(-1, 0)$ with $(1, 0)$ such that $L(c) = 2$.

Note that the curve c in Example 17 is distance realising.

Definition 2.16. *Let (X, d) be a metric space. A subset $A \subset X$ is* compact *if for every sequence $x_n \in A$ there exists a subsequence $x_{n_j} \in A$ and an $x_\infty \in A$ such that*

$$d(x_{n_j}, x_\infty) \to 0.$$

The space X is called complete*, if every Cauchy sequence in X is convergent, i.e., for every sequence $x_n \in X$ with*

$$\forall \epsilon > 0 \, \exists N_\epsilon \geq 0 : \quad d(x_n, x_m) < \epsilon \quad \forall n, m \geq N_\epsilon$$

there exists an $x_\infty \in X$ such that

$$d(x_n, x_\infty) \to 0.$$

Example 19 ($\mathbb{R} \setminus \{0\}$ is not complete)**.** $\mathbb{R} \setminus \{0\}$ *with the standard distance function $d(x, y) = |x - y|$ is* not complete*, since $x_n = \frac{1}{n}$ is a Cauchy sequence but its limit* 0 *is not in* $\mathbb{R} \setminus \{0\}$.

3 Levi-Civita connection and parallel transport

We have already seen how to differentiate a function $f : M \to \mathbb{R}$ along a curve $c : (-\epsilon, \epsilon) \to M$, namely, the directional derivative of f along c is given by

$$c'(0)(f) := (f \circ c)'(0),$$

and the map $c'(0) : D(M, c(0)) \to \mathbb{R}$ is called a tangent vector. We also introduced vector fields $X : M \to TM$, where $X(p) \in T_pM$. Vector fields can be considered as functions on M, but with a more complicated range: no longer $\mathbb{R}$ but the space TM.

Our aim in this section is to introduce a directional derivative of a vector field $X : M \to TM$ along a curve $c : (-\epsilon, \epsilon) \to M$ with particular properties. This will lead to the so-called *covariant derivative* or *Levi-Civita connection*.

3.1 Covariant derivative and Levi-Civita connection

As mentioned previously, we want to introduce a derivative of a vector field in the direction of a tangent vector, the so-called *covariant derivative*. To do this in a *unique* way on a manifold X, we need a Riemannian metric g. Note that such a metric is not needed for the directional derivative of a function. We first discuss the easiest case, $M = \mathbb{R}^n$ with the standard Euclidean metric $\langle \cdot, \cdot \rangle$.

Example 20 (Directional derivative of a vector field in $\mathbb{R}^n$)**.** *By the identification $T_p\mathbb{R}^n \cong \mathbb{R}^n$, a vector field X on $\mathbb{R}^n$ can be considered as a differentiable map $X : \mathbb{R}^n \to \mathbb{R}^n$, i.e,*

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}|_p \cong \begin{pmatrix} a_1(p) \\ \vdots \\ a_n(p) \end{pmatrix}.$$

For a tangent vector $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, we can naturally define the derivative of X in the direction v by

$$\begin{aligned}
\nabla_v X \quad &:= \quad DX(p)(v) = \lim_{t\to 0} \frac{X(p+tv) - X(p)}{t} = \begin{pmatrix} \lim_{t\to 0} \frac{a_1(p+tv)-a_1(p)}{t} \\ \vdots \\ \lim_{t\to 0} \frac{a_n(p+tv)-a_n(p)}{t} \end{pmatrix} \\
&= \begin{pmatrix} v(a_1) \\ \vdots \\ v(a_n) \end{pmatrix} = \sum_{i=1}^n v(a_i) e_i \cong \sum_{i=1}^n v(a_i) \frac{\partial}{\partial x_i}|_p \in T_p\mathbb{R}^n.
\end{aligned}$$

We call $\nabla_v X$ the covariant derivative *of X in the direction v.*

Next, let us look at *properties* of the construction $\nabla_v X$ in the example: Recall that $v \in T_p\mathbb{R}^n \cong \mathbb{R}^n$:

(a) We have for $X, Y : \mathbb{R}^n \to \mathbb{R}^n$:

$$\nabla_v(X+Y) = \nabla_v X + \nabla_v Y.$$

Let $X = (a_1, \ldots, a_n)^\top$, $Y = (b_1, \ldots, b_n)^\top$. Then

$$\nabla_v(X+Y) = \sum_{i=1}^n v(a_i + b_i) e_i = \sum_{i=1}^n v(a_i) e_i + \sum_{i=1}^n v(b_i) e_i = \nabla_v X + \nabla_v Y.$$

(b) We have the following type of "product rule" for $X : \mathbb{R}^n \to \mathbb{R}^n$ and $f \in C^\infty(\mathbb{R}^n)$:

$$\nabla_v(fX) = v(f)X(p) + f(p)\nabla_v X.$$

With X as above, we obtain

$$\begin{aligned}
\nabla_v(fX) = \sum_{i=1}^n v(fa_i) e_i = \sum_{i=1}^n f(p) v(a_i) e_i + \sum_{i=1}^n a_i(p) v(f) e_i& \\
= f(p)\nabla_v X + v(f) X(p)&.
\end{aligned}$$

(c) We have for $v, w \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$:

$$\nabla_{\alpha v + \beta w} X = \alpha \nabla_v X + \beta \nabla_v X.$$

This is checked in the following way:

$$\begin{aligned}
\nabla_{\alpha v + \beta w} X = \sum_{i=1}^n (\alpha v + \beta w)(a_i) e_i = \alpha \sum_{i=1}^n v(a_i) e_i + \beta \sum_{i=1}^n w(a_i) e_i& \\
= \alpha \nabla_v X + \beta \nabla_w X&.
\end{aligned}$$

(d) For two vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$, the inner product $\langle X, Y\rangle$ is a function in $C^\infty(\mathbb{R}^n)$. We have

$$v(\langle X, Y\rangle) = \langle \nabla_v X, Y(p)\rangle + \langle X(p), \nabla_v Y\rangle. \tag{7}$$

To see this, we assume X, Y to be as above and first calculate

$$\langle \nabla_v X, Y(p) \rangle = \langle \sum_{i=1}^n v(a_i) e_i, \sum_{j=1}^n b_j e_j \rangle = \sum_{i=1}^n v(a_i) b_i(p).$$

This implies

$$\begin{aligned} \langle \nabla_v X, Y(p) \rangle + \langle X(p), \nabla_v Y \rangle &= \sum_{i=1}^n v(a_i) b_i(p) + v(b_i) a_i(p) \\ &= \sum_{i=1}^n v(a_i b_i) = v(\sum_{i=1}^n a_i b_i) = v(\langle X, Y \rangle). \end{aligned}$$

Property (7) is called the "Riemannian" property of the covariant derivative $\nabla_v S$.

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To derive the last important property we need to define $\nabla_X Y$ for two vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$. Let $X = \sum a_i \frac{\partial}{\partial x_i}$ and $Y = \sum b_j \frac{\partial}{\partial x_j}$. We define

$$(\nabla_X Y)(p) := \nabla_{X(p)} Y \in \mathbb{R}^n \cong T_p \mathbb{R}^n,$$

which makes $\nabla_X Y$, again, to a vector field. We have, using the identification $\mathbb{R}^n \cong T_p \mathbb{R}^n$

$$(\nabla_X Y)(p) = \sum_{i=1}^n X(p)(b_i) e_i \cong \sum_{i=1}^n \left(\sum_{j=1}^n a_j(p) \frac{\partial b_i}{\partial x_j}(p) \right) \frac{\partial}{\partial x_i}|_p.$$

This implies that

$$(\nabla_X Y)(p) - (\nabla_Y X)(p) = \sum_{i=1}^n \left(\sum_{j=1}^n a_j(p) \frac{\partial b_i}{\partial x_j}(p) - b_j(p) \frac{\partial a_i}{\partial x_j}(p) \right) \frac{\partial}{\partial x_i}|_p = [X, Y](p),$$

where $[X, Y]$ denotes the Lie bracket, introduced in Section 1.4. We call the map $(X, Y) \mapsto \nabla_X Y - \nabla_Y X - [X, Y]$ the *torsion* of the covariant derivative ∇ and say that ∇ is *torsion free*.

Repetition: For a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ and a tangent vector $v \in T_p \mathbb{R}^n \cong \mathbb{R}^n$ we define

$$\nabla_v X := \lim_{t \to 0} \frac{X(p + tv) - X(p)}{t} \in \mathbb{R}^n \cong T_p \mathbb{R}^n.$$

Moreover, for a pair of vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$, we define

$$(\nabla_Y X)(p) = \nabla_{Y(p)} X = \lim_{t \to 0} \frac{X(p + tY(p)) - X(p)}{t} \in \mathbb{R}^n \cong T_p \mathbb{R}^n.$$

Let $\mathcal{X}(\mathbb{R}^n)$ denote the $\mathbb{R}$-vector space of all vector fields on $\mathbb{R}^n$. Then the map $\nabla : \mathcal{X}(\mathbb{R}^n) \times \mathcal{X}(\mathbb{R}^n) \to \mathcal{X}(\mathbb{R}^n)$, $(X, Y) \mapsto \nabla_X Y$ satisfies

(a) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(b) $\nabla_X(fY) = (Xf)\cdot Y + f\cdot \nabla_X Y$ for $f \in C^\infty(\mathbb{R}^n)$,

(c) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$, for $f, g \in C^\infty(\mathbb{R}^n)$,

(d) Riemannian: $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,

(e) torison free: $\nabla_X Y - \nabla_Y X = [X, Y]$.

Next, we want to transfer the concept of a covariant derivative of a vector field to manifolds:

Definition 3.1. *Let M be a differentiable manifold and $\mathcal{X}(M)$ be the space of all vector fields on M. A map*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M), \quad (X, Y) \mapsto \nabla_X Y,$$

is called covariant derivative *or* affine connection*, if the above properties (a), (b) and (c) are satisfied.*

Theorem 3.2 (Levi-Civita)**.** *Let (M, g) be a Riemannian manifold. Then there exists a* unique *covariant derivative ∇ on M, which is Riemannian and torsion free. This covariant derivative is called the* Levi-Civita connection *of M.*

Proof. Uniqueness: Let ∇ be satisfying (a)-(e). Let $X, Y, Z \in \mathcal{X}(M)$. Since ∇ is Riemannian, we have

$$\begin{aligned} X(\langle Y, Z\rangle) &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, , && (8)\\ Y(\langle Z, X\rangle) &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle, , && (9)\\ Z(\langle X, Y\rangle) &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle, . && (10)\end{aligned}$$

Now we look at (8)+(9)−(10):

$$\begin{aligned} X(\langle Y, Z\rangle) + Y(\langle Z, X\rangle) - Z(\langle X, Y\rangle) = \langle \nabla_X Y, Z\rangle &+ (\langle Y, \nabla_X Z\rangle - \langle \nabla_Z X, Y\rangle)\\ + (\langle \nabla_Y Z, X\rangle - \langle X, \nabla_Z Y\rangle) &+ \underbrace{\langle Z, \nabla_Y X\rangle}_{=\langle Z,[Y,X]\rangle + \langle \nabla_X Y, Z\rangle} \quad . \end{aligned}$$

This implies that

$$\begin{aligned} &\langle \nabla_X Y, Z\rangle =\\ &\frac{1}{2}\left(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle\right). \end{aligned} \tag{11}$$

Now, if $\widetilde{\nabla}_X Y$ were another connection with the properties (a)-(e), then we would have for every $p \in M$:

$$\langle \underbrace{\widetilde{\nabla}_X Y(p) - \nabla_X Y(p)}_{=:w}, v\rangle = 0, \text{ for all } v \in T_pM,$$

and thus, choosing $v = w$,

$$\|\widetilde{\nabla}_X Y(p) - \nabla_X Y(p)\|_p^2 = 0,$$

i.e., $\widetilde{\nabla}_X Y(p) = \nabla_X Y(p)$. This proves uniqueness.

Existence: The right hand side of (11) is linear in Z, so $\nabla_X Y$ is well-defined by (11). Properties (a) and (c) are easy to check. For Property (b) we use Proposition 1.17 and conclude that

$$\begin{aligned}
&\langle \nabla_X fY, Z\rangle = \\
&f\langle \nabla_X Y, Z\rangle + \frac{1}{2}\left((Xf)\langle Y, Z\rangle - (Zf)\langle X, Y\rangle - \langle X, -Z(f)Y\rangle + \langle Z, (Xf)Y\rangle\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \langle f\nabla_X Y + X(f)Y, Z\rangle.
\end{aligned}$$

We also conclude from (11) that

$$\langle \nabla_X Y - \nabla_Y X, Z\rangle = \frac{1}{2}\left(\langle Z, [X,Y]\rangle - \langle Z, [Y,X]\rangle\right) = \langle [X,Y], Z\rangle,$$

which proves (e), i.e., that ∇ is torsion free. Finally, we conclude from (11) that

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \frac{1}{2}\left(X\langle Y, Z\rangle + X\langle Z, Y\rangle\right) = X\langle Y, Z\rangle,$$

which proves (d), i.e., that ∇ is Riemannian. $\square$

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Example 21 (Levi-Civita connection for a surface in $\mathbb{R}^3$)**.** *Let $M \subset \mathbb{R}^3$ be a surface with Riemannian metric g induced from the standard inner product of the ambient space $\mathbb{R}^3$ and $\widetilde{\nabla}$ be the Levi-Civita connection of $\mathbb{R}^3$, i.e.,*

$$\widetilde{\nabla}_v X = \lim_{t \to 0} \frac{X(p+tv) - X(p)}{t}$$

for $v \in T_p\mathbb{R}^3$. For every $p \in M$ we have a canonical orthogonal projection

$$\pi_p : T_p\mathbb{R}^3 \to T_pM.$$

Let $X \in \mathcal{X}(M)$ and $\widetilde{X} \in \mathcal{X}(\mathbb{R}^3)$ be an extension of X, i.e., $\widetilde{X}(p) = X(p)$ for all $p \in M$. Then we define for $v \in T_pM \subset T_p\mathbb{R}^3$:

$$\nabla_v X := \pi_p\left(\widetilde{\nabla}_v \widetilde{X}\right). \tag{12}$$

It can be checked that this definition does not depend on the choice of extension $\widetilde{X}$, and that ∇ satisfies all conditions (a)-(e). This means that ∇, defined in (12)*, is the Levi-Civita connection of M.*

Remark. *The previous example generalises to arbitrary submanifolds $M \subset \mathbb{R}^n$ with the induced Riemannian metric on M.*

3.2 Calculation of Christoffel symbols

Definition 3.3. *Let (M, g) be a Riemannian manifold and ∇ its Levi-Civita connection. Let $\varphi : U \to V \subset \mathbb{R}^n$, $U \subset M$ be a coordinate chart with coordinate functions $x_1, \dots, x_n : U \to \mathbb{R}$. Then we have*

$$\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)(p) \in T_pM,$$

i.e., there exist uniquely defined $\Gamma_{ij}^k(p) \in \mathbb{R}$ *with*

$$\left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)(p) = \sum_{k=1}^n \Gamma_{ij}^k(p) \frac{\partial}{\partial x_k}|_p.$$

The functions $\Gamma_{ij}^k : U \to \mathbb{R}$ *are called the* Christoffel symbols *of the connection* ∇ *with respect to the coordinate chart* φ*.*

The above formula (11) enables us to calculate Γ_{ij}^k directly from the functions $g_{ij} : U \to \mathbb{R}$,

$$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p\right).$$

Note that

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$

since

$$\begin{aligned}\left(\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]\right)(f)(p) &= \frac{\partial}{\partial x_i}|_p \frac{\partial(f \circ \varphi^{-1})}{\partial x_j} \circ \varphi - \frac{\partial}{\partial x_j}|_p \frac{\partial(f \circ \varphi^{-1})}{\partial x_i} \circ \varphi \\ &= \frac{\partial^2(f \circ \varphi^{-1})}{\partial x_i \partial x_j}(\varphi(p)) - \frac{\partial^2(f \circ \varphi^{-1})}{\partial x_i j \partial x_i}(\varphi(p)) = 0.\end{aligned}$$

So (11) simplifies to

$$\begin{aligned}&\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l} \rangle = \\ &\quad \frac{1}{2}\left(\frac{\partial}{\partial x_i}\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l}\rangle + \frac{\partial}{\partial x_j}\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_l}\rangle - \frac{\partial}{\partial x_l}\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle\right) = \\ &\qquad \frac{1}{2}\left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij}\right).\end{aligned}$$

On the other hand

$$\langle \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l} \rangle = \langle \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \rangle = \sum_k \Gamma_{ij}^k g_{kl}.$$

This yields

$$\sum_k \Gamma_{ij}^k g_{kl} = \frac{1}{2}(g_{jl,i} + g_{il,j} - g_{ij,l}),$$

where $g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab}$. $(g_{ij}(p))_{1 \le i,j \le n}$ is an invertible matrix and we denote the entries of $(g_{ij}(p))^{-1} \in GL(n, \mathbb{R})$ by $g^{ij}(p)$, i.e.,

$$\left(g^{ij}(p)\right) = (g_{ij}(p))^{-1} \in GL(n, \mathbb{R}).$$

This yields

$$\Gamma_{ij}^m = \sum_k \Gamma_{ij}^k \delta_{km} = \sum_{k,l} \Gamma_{ij}^k (g_{kl} g^{lm}) = \frac{1}{2} \sum_l g^{lm}(g_{il,j} + g_{jl,i} - g_{ij,l}).$$

So we have proved

Proposition 3.4. *The Christoffel symbols, given by*

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k},$$

can be explicitely calculated by the formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}).$$

As an immediate consequence of this proposition we have

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

Example 22 (Christoffel symbols of $\mathbb{H}^2$)**.** *We have*

$$(g_{ij}(z)) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}, \qquad (g^{ij}(z)) = \begin{pmatrix} y^2 & 0 \\ 0 & y^2 \end{pmatrix}.$$

This implies that

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2} g^{11}(g_{11,1} + g_{11,1} - g_{11,1}) = 0, \\
\Gamma_{11}^2 &= \frac{1}{2} g^{22}(g_{12,1} + g_{12,1} - g_{11,2}) = \frac{1}{2} y^2(-\frac{-2}{y^3}) = \frac{1}{y}, \\
\Gamma_{12}^1 &= \frac{1}{2} g^{11}(g_{11,2} + g_{21,1} - g_{12,1}) = \frac{1}{2} y^2(\frac{-2}{y^3}) = -\frac{1}{y} = \Gamma_{21}^1, \\
\Gamma_{12}^2 &= \frac{1}{2} g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) = 0 = \Gamma_{21}^2, \\
\Gamma_{22}^1 &= \frac{1}{2} g^{11}(g_{21,2} + g_{21,2} - g_{22,1}) = 0, \\
\Gamma_{22}^2 &= \frac{1}{2} g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = \frac{1}{2} y^2 \frac{-2}{y^3} = -\frac{1}{y}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} &= \Gamma_{11}^1 \frac{\partial}{\partial x} + \Gamma_{11}^2 \frac{\partial}{\partial y} = \frac{1}{y} \frac{\partial}{\partial y}, \\
\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} &= \Gamma_{12}^1 \frac{\partial}{\partial x} + \Gamma_{12}^2 \frac{\partial}{\partial y} = -\frac{1}{y} \frac{\partial}{\partial x}, \\
\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} &= \underbrace{\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]}_{=0} + \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = -\frac{1}{y} \frac{\partial}{\partial x}, \\
\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} &= \Gamma_{22}^1 \frac{\partial}{\partial x} + \Gamma_{22}^2 \frac{\partial}{\partial y} = -\frac{1}{y} \frac{\partial}{\partial y}.
\end{aligned}$$

Let $X(z = x + iy) = y \frac{\partial}{\partial y}|_z$. Then $\|X(z)\|_z = 1$. What is $\nabla_X X$? We have

$$(\nabla_X X)(z) = y \nabla_{\frac{\partial}{\partial y}|_z} \left(y \frac{\partial}{\partial y} \right) = y \frac{\partial}{\partial y}|_z + y^2 \nabla_{\frac{\partial}{\partial y}|_z} \frac{\partial}{\partial y} = y \frac{\partial}{\partial y}|_z + y^2 \left(-\frac{1}{y} \right) \frac{\partial}{\partial y}|_z = 0,$$

so the covariant derivative of X in the direction of X vanishes.

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3.3 Parallel transport

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $c : [a, b] \to M$ be a differentiable curve and $X \in \mathcal{X}(M)$ be a vector field. Then $\nabla_{c'(t)} X$ depends only on the vector field X along c, i.e., if $\widetilde{X} \in \mathcal{X}(M)$ is another vector field with

$$\widetilde{X}(c(t)) = X(c(t)),$$

then $\nabla_{c'(t)} X = \nabla_{c'(t)} \widetilde{X}$. This observation leads us to introduce vector fields along curves:

Definition 3.5. *Let $c : [a, b] \to M$ be a differentiable curve. A map*

$$X : [a, b] \to TM$$

satisfying $X(t) \in T_{c(t)}M$ is called a vector field along the curve c. *We denote the vector space of all vector fields along the curve c by $\mathcal{X}_c(M)$.*

Example 23 ($c'(t)$ as a vector field along itself)**.** *Let $c : [a, b] \to TM$, $t \mapsto c'(t) \in T_{c(t)}M$ is a vector field along c.*

Proposition 3.6. *Let ∇ be the Levi-Civita connection of a Riemannian manifold (M, g). Let $c : [a, b] \to M$ be a differentiable curve. Then there exists a* unique *map*

$$\frac{D}{dt} : \mathcal{X}_c(M) \to \mathcal{X}_c(M)$$

with the following properties:

(a) $\frac{D}{dt}(X + Y) = \frac{D}{dt}X + \frac{D}{dt}Y$ *for all $X, Y \in \mathcal{X}_c(M)$,*

(b) $\frac{D}{dt}(fX) = f'X + f\frac{D}{dt}X$ *for all $X \in \mathcal{X}_c(M)$ and differentiable $f : [a, b] \to M$,*

(c) If $\widetilde{X} \in \mathcal{X}(M)$ is a local extension of X, i.e., there are $t_0 \in [a, b]$ and $\epsilon > 0$ such that

$$\widetilde{X}(c(t)) = X(t) \quad \textit{for all } t \in [t_0 - \epsilon, t_0 + \epsilon],$$

then we have

$$\left(\frac{D}{dt}X\right)(t_0) = \nabla_{c'(t_0)} \widetilde{X}.$$

The map $\frac{D}{dt}$ is called the covariant derivative along the curve c.

Proof. Uniqueness: Let $\varphi = (x_1, \dots, x_n) : U \to V \subset \mathbb{R}^n$ be a coordinate chart with $c([t_0 - \epsilon, t_0 + \epsilon]) \subset U$. Let $X \in \mathcal{X}_c(M)$. Then we can write

$$X(t) = \sum_{i=1}^n a_i(t) \frac{\partial}{\partial x_i}|_{c(t)}.$$

Using (a) and (b), we conclude that

$$\frac{D}{dt}X = \sum_{i=1}^n a_i' \left(\frac{\partial}{\partial x_i} \circ c\right) + a_i \frac{D}{dt}\left(\frac{\partial}{\partial x_i} \circ c\right).$$

Finally, using (c) we obtain

$$\left(\frac{D}{dt}X\right)(t) = \sum_{i=1}^{n} a_i'(t)\frac{\partial}{\partial x_i}|_{c(t)} + a_i(t)\nabla_{c'(t)}\frac{\partial}{\partial x_i} \in T_{c(t)}M. \tag{13}$$

So $\frac{D}{dt}X$ is uniquely determined by (13).

Existence: The definition (13) for $\frac{D}{dt}X$ with $X = \sum a_i \left(\frac{\partial}{\partial x_i} \circ c\right)$ yields a vector field along c and one can check that properties (a), (b) and (c) are satisfied. □

Example 24 ($\frac{D}{dt}$ for surfaces in $\mathbb{R}^3$)**.** *let $M \subset \mathbb{R}^3$ be a surface. Let $\frac{D}{dt}$ denote the covariant derivative along a curve $c : [a,b] \to M$. Let $X : [a,b] \to TM$ be a vector field along c, i.e.,*

$$X(t) \in T_{c(t)}M \subset T_{c(t)}\mathbb{R}^3 \cong \mathbb{R}^3.$$

Let $X(t) = (a_1(t), a_2(t), a_3(t))^\top \in \mathbb{R}^3$. We conclude from Example 21 that

$$\left(\frac{D}{dt}X\right)(t) = \pi_{c(t)}\left((a_1'(t), a_2'(t), a_3'(t))^\top\right) \in T_{c(t)}M,$$

where $\pi_p : \mathbb{R}^3 \to T_pM$ for $p \in M$ is the orthogonal projection. In particular, if $c(t) = (c_1(t), c_2(t), c_3(t)) \in M$, then

$$c'(t) = (c_1'(t), c_2'(t), c_3'(t))^\top \in T_{c(t)}M$$

and

$$\left(\frac{D}{dt}c'\right)(t) = \pi_{c(t)}\left((c_1''(t), c_2''(t), c_3''(t))^\top\right) = \pi_{c(t)}(c''(t)),$$

and $\frac{D}{dt}c' \equiv 0$ is equivalent to $c''(t) \perp T_{c(t)}M$ for all $t \in [a,b]$. But this, in turn, is precisely the condition for c to be a geodesic of the surface (cf. Differential Geometry III).

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Definition 3.7. *let $c : [a,b] \to M$ be a curve in a Riemannian manifold (M,g) and $\frac{D}{dt}$ the covariant derivative along c. A vector field $X : [a,b] \to TM$ is called* parallel along *c, if we have*

$$\frac{D}{dt}X \equiv 0.$$

Example 25 (Parallel vector fields in $\mathbb{R}^2$)**.** *Let $\mathbb{R}^2$ be a Riemannian manifold with the standard Euclidean Riemannian metric and $c : [a,b] \to \mathbb{R}^2$ be a curve. For a vector field $X : [a,b] \to \mathbb{R}^2$ along c with $X(t) = (a_1(t), a_2(t))$ we have*

$$\left(\frac{D}{dt}X\right)(t) = (a_1'(t), a_2'(t)).$$

So $\frac{D}{dt}X \equiv 0$ means that a_1, a_2 are both constant functions.

This example explains why a vector field X is called parallel if $\frac{D}{dt}X \equiv 0$. In $\mathbb{R}^2$ this means that the vector field is constant, i.e., consists of parallel vectors along c.

Parallel vector fields along a curve $c : [a,b] \to M$ are uniquely determined by a start vector $v \in T_{c(a)}M$:

Theorem 3.8. *Let $c : [a,b] \to M$ be a curve in a Riemannian manifold (M,g) and $v \in T_{c(a)}M$. Then there exists a* unique *parallel vector field $X \in \mathcal{X}_c(M)$ with*

$$X(a) = v \in T_{c(a)}M.$$

If $n = \dim M$, then the space of all parallel vector fields in $\mathcal{X}_c(M)$ is a n-dimensional vector space over $\mathbb{R}$.

Proof. Assume, for simplicity, that there exists a coordinate chart $\varphi := (x_1,\dots,x_n) : U \to V \subset \mathbb{R}^n$ with $c([a,b] \subset U \subset M$. For a vector field $X \in \mathcal{X}_c(M)$ with $X(t) = \sum a_i(t)\frac{\partial}{\partial x_i}|_{c(t)}$, we have by (13):

$$\left(\frac{D}{dt}X\right)(t) = \sum_i a_i'(t)\frac{\partial}{\partial x_i}|_{c(t)} + a_i(t)\nabla_{c'(t)}\frac{\partial}{\partial x_i}.$$

Let $\varphi \circ c =: (c_1,\dots,c_n) : [a.b] \to \mathbb{R}^n$. Then $c'(t) = \sum_j c_j'(t)\frac{\partial}{\partial x_j}|_{c(t)}$ and we obtain

$$\begin{aligned}
\left(\frac{D}{dt}X\right)(t) &= \sum_i \left(a_i'(t)\frac{\partial}{\partial x_i}|_{c(t)} + a_i(t)\sum_j c_j'(t)\nabla_{\frac{\partial}{\partial x_j}|_{c(t)}}\frac{\partial}{\partial x_i}\right) \\
&= \sum_i \left(a_i'(t)\frac{\partial}{\partial x_i}|_{c(t)} + a_i(t)\sum_{j,k} c_j'(t)\Gamma_{ij}^k(c(t))\frac{\partial}{\partial x_k}|_{c(t)}\right) \\
&= \sum_k \left(a_k'(t) + \sum_{i,j} a_i(t)c_j'(t)\Gamma_{ij}^k(c(t))\right)\frac{\partial}{\partial x_k}|_{c(t)}.
\end{aligned}$$

So $X = \sum a_i\left(\frac{\partial}{\partial x_i}\circ c\right)$ is parallel along c if and only if

$$a_k'(t) + \sum_i \left(\sum_j c_j'(t)\Gamma_{ij}^k(c(t))\right) a_i(t) = 0 \qquad \text{for all } t \text{ and } k.$$

This is a vector-valued homogeneous linear ordinary differential equation, since it can be rewritten as

$$\begin{pmatrix} a_1'(t) \\ \vdots \\ a_n'(t) \end{pmatrix} = A(t)\begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix}, \tag{14}$$

with $A(t) = (A_{ki}(t))$ and

$$A_{ki}(t) = -\sum_j c_j'(t)\Gamma_{ij}^k(c(t)).$$

The theory of ordinary differential equations tells us that every initial choice $(a_1(a),\dots,a_n(a)) = (\alpha_1,\dots,\alpha_n)$ yields a unique solution of (14). For $v =$

$\sum_i \alpha_i \frac{\partial}{\partial x_i}|_{c(a)}$, this proves existence and uniqueness of the parallel vector field $X = \sum a_i \left(\frac{\partial}{\partial x_i} \circ c\right)$ along c with $X(a) = v$.

Note that the set of parallel vector fields along c forms a R-vector space: if $X, Y \in \mathcal{X}_c(M)$ with $\frac{D}{dt}X = 0$ and $\frac{D}{dt}Y = 0$ and $a, b \in \mathbb{R}$, then we have

$$\frac{D}{dt}(aX + bY) = a\frac{D}{dt}X + b\frac{D}{dt}Y = 0.$$

Since every parallel vector field along c is uniquely determined by its value $X(a) \in T_{c(a)}M$, we have

$$\dim\left\{X \in \mathcal{X}_c(M) \mid \frac{D}{dt}X = 0\right\} = \dim T_{c(a)}M = n.$$

□

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Example 26 (Parallel vector fields in $\mathbb{H}^2$). *We know from Example 22 that*

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{22} = 0$$

and

$$\Gamma^2_{11} = \frac{1}{y}, \quad \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\frac{1}{y}.$$

Let $c(t) = t + yi$.

We want to find the parallel vector field $X(t) = a_1(t)\frac{\partial}{\partial x}|_{c(t)} + a_2(t)\frac{\partial}{\partial y}|_{c(t)}$ along c with $X(0) = \frac{\partial}{\partial y}|_{yi}$. Note that $c'(t) = 1$, so $c_1'(t) = 1$ and $c_2'(t) = 0$. So equation (14) assumes the concrete form

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = -\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} \\ \Gamma^2_{11} & \Gamma^2_{21} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that the involved matrix is constant, since y stays fixed along the curve $c(t)$. Therefore the solution is given by

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \exp\left(t\begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}^k\right)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}^{2k} = \begin{pmatrix} -\frac{1}{y^2} & 0 \\ 0 & -\frac{1}{y^2} \end{pmatrix}^k = \begin{pmatrix} \frac{(-1)^k}{y^{2k}} & 0 \\ 0 & \frac{(-1)^k}{y^{2k}} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}^{2k+1} = \begin{pmatrix} \frac{(-1)^k}{y^{2k}} & 0 \\ 0 & \frac{(-1)^k}{y^{2k}} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{(-1)^k}{y^{2k+1}} \\ \frac{(-1)^{k+1}}{y^{2k+1}} & 0 \end{pmatrix}.$$

This means that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{t^k}{k!}\begin{pmatrix} 0 & \frac{1}{y} \\ -\frac{1}{y} & 0 \end{pmatrix}^k &= \begin{pmatrix} \sum_k \frac{t^{2k}}{(2k)!}\frac{(-1)^k}{y^{2k}} & \sum_k \frac{t^{2k+1}}{(2k+1)!}\frac{(-1)^k}{y^{2k+1}} \\ -\sum_k \frac{t^{2k+1}}{(2k+1)!}\frac{(-1)^k}{y^{2k+1}} & \sum_k \frac{t^{2k}}{(2k)!}\frac{(-1)^k}{y^{2k}} \end{pmatrix} \\ &= \begin{pmatrix} \cos(t/y) & \sin(t/y) \\ -\sin(t/y) & \cos(t/y) \end{pmatrix}, \end{aligned}$$

and, consequently,

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t/y) & \sin(t/y) \\ -\sin(t/y) & \cos(t/y) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin(t/y) \\ \cos(t/y) \end{pmatrix}.$$

So the parallel vector field X along c with $X(0) = \frac{\partial}{\partial y}|_{yi}$ is given by

$$X(t) = \sin(t/y)\frac{\partial}{\partial x}|_{t+yi} + \cos(t/y)\frac{\partial}{\partial y}|_{t+yi},$$

i.e., X rotates clockwise as we move horizontally to the right along c.

Definition 3.9. *Let (M, g) be a Riemannian manifold and $c : [a, b] \to M$ be a curve. The* parallel transport *P_c is a linear map*

$$P_c : T_{c(a)}M \to T_{c(b)}M,$$

which is defined as follows:

$$P_c(v) = X(b),$$

where $v \in T_{c(a)}M$ and $X \in \mathcal{X}_c(M)$ is the unique parallel vector field along c with $X(a) = v$.

Remarks. *(a) Note that the isomorphism $P_c : T_{c(a)}M \to T_{c(b)}M$ depends in general on the curve c, i.e., if $\gamma : [\alpha, \beta] \to M$ is another curve with $\gamma(\alpha) = c(a)$ and $\gamma(\beta) = c(b)$, then we do not necessarily have $P_c = P_\gamma : T_{c(a)}M \to T_{c(b)}M$.*

(b) The parallel transport along a curve $c : [a, b] \to M$ yields a linear isomorphism $P_c : T_pM \to T_qM$ with $p = c(a)$, $q = c(b)$, i.e., a connection *between the two disjoint tangent spaces T_pM and T_qM. Note that parallel transport is defined via the Levi-Civita connection.Hence: The* Levi-Civita connection *induces a* connection *between disjoint tangent spaces along curves connecting their footpoints. This is the motivation for the word "connection".*

Proposition 3.10. *The parallel transport*

$$P_c : T_{c(a)}M \to T_{c(b)}M$$

is a linear isometry*, i.e., we have*

$$g_{c(a)}(v_1, v_2) = g_{c(b)}(P_c(v_1), P_c(v_2)).$$

The proof of this fact is Exercise 16.