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Riemannian Geometry IV

Solutions, set 1.

Exercise 1. Every line in $\mathbb{R}^3$ through the origin is of the form $L = \mathbb{R}(p_1, p_2, p_3)$, where $(p_1, p_2, p_3) \neq 0$. This means that we have $p_j \neq 0$ for at least one $j \in \{1, 2, 3\}$. Then we obviously have $L \in U_j$. The inverse maps are given by

$$\varphi_1^{-1}(x_1, x_2) = \mathbb{R} \cdot (1, x_1, x_2), \quad \varphi_2^{-1}(x_1, x_2) = \mathbb{R} \cdot (x_1, 1, x_2),$$
$$\varphi_3^{-1}(x_1, x_2) = \mathbb{R} \cdot (x_1, x_2, 1).$$

The coordinate change $\varphi_1 \circ \varphi_2^{-1}$ is a map from $\{(x_1, x_2) \mid x_1 \neq 0\} \subset \mathbb{R}^2$ to $\{(x_1, x_2) \mid x_1 \neq 0\} \subset \mathbb{R}^2$ and is given by

$$\varphi_1 \circ \varphi_2^{-1}(x_1, x_2) = \varphi_1(\mathbb{R}(x_1, 1, x_2)) = \left(\frac{1}{x_1}, \frac{x_2}{x_1} \right).$$

A similar description holds for all other coordinate changes. Since we excluded $x_1 = 0$, the map

$$(x_1, x_2) \mapsto \left(\frac{1}{x_1}, \frac{x_2}{x_1} \right)$$

is obviously (infinitely often) differentiable in both x_1 and x_2.

Exercise 2.

(a) Let $g(\lambda) = f(\lambda x_1, \dots, \lambda x_k) = \lambda^m f(x_1, \dots, x_k)$. Using the chain rule, we obtain

$$\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(\lambda x) x_i = m\lambda^{m-1} f(x_1, \dots, x_k).$$

Choosing $\lambda = 1$, the left side is equal to $\langle \mathrm{grad} f(x), x \rangle$ and the right side is $mf(x)$. This finishes the proof of (a).

(b) Let f be a homogeneous polynomial of degree $m \geq 1$ and $y \neq 0$. Let $x \in f^{-1}(y)$. Then we obtain with (a):

$$\langle \mathrm{grad} f(x), x \rangle = m f(y) \neq 0.$$

This implies that $\mathrm{grad} f(x) \neq 0$, so $Df(x) : \mathbb{R}^k \to \mathbb{R}$ is surjective for all $x \in f^{-1}(y)$. Therefore, $y \neq 0$ is a regular value.

(c) The group $SL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$ ist equal to $f^{-1}(1)$, where $f(A) = \det A$. Now, f is a homogeneous polynomial of degree n in $\mathbb{R}^{n^2}$, so 1 is a regular value of f, by (b). Theorem 1.5 implies that $SL(n, \mathbb{R}) = f^{-1}(1)$ is a differentiable manifold of dimension $n^2 - 1$.