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# Riemannian Geometry IV

**Solutions, set 2.**

**Exercise 3.** We have

$$\frac{\partial}{\partial x_i}\Big|_p(f) = \frac{\partial(f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) = \frac{\partial}{\partial x_i}(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1})(\varphi(p)).$$

The last expression above is partial derivative in coordinate direction $x_i$ of the composition of the two functions $\psi\circ\varphi^{-1}: V_1\subset\mathbb{R}^n\to V_2\subset\mathbb{R}^n$ and $f\circ\psi^{-1}: V_2\subset\mathbb{R}^n\to\mathbb{R}$. The chain rule tells us that

$$\frac{\partial}{\partial x_i}(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1})(\varphi(p)) = \sum_{j=1}^n \frac{\partial(f\circ\psi^{-1})}{\partial y_j}(\psi(p))\cdot\frac{\partial(y_j\circ\varphi^{-1})}{\partial x_i}(\varphi(p)).$$

Here $\frac{\partial}{\partial y_j}$ denotes partial derivative in the $j$-th coordinate direction of $V_2\subset\mathbb{R}^n$ and $y_j$ in the expression $y_j\circ\varphi^{-1}$ denotes the $j$-th component function of the map $\psi$. So we finally end up with

$$\frac{\partial}{\partial x_i}|_p(f) = \sum_{j=1}^n \frac{\partial(y_j\circ\varphi^{-1})}{\partial x_i}(\varphi(p))\cdot\frac{\partial}{\partial y_j}|_p(f).$$

**Exercise 4.** Let $A: (-\epsilon,\epsilon)\to SO(n)$ be a differentiable curve on the differentiable manifold $SO(n)$ with $A(0)=I$. Then we know that

$$A(t)(A(t))^\top = I,$$

for all $t\in(-\epsilon,\epsilon)$. Differentiation gives

$$A'(0)(A(0))^\top + A(0)(A'(0))^\top = A'(0)I^\top + I(A'(0))^\top = A'(0)+(A'(0))^\top = 0.$$

So we conclude that

$$T_I SO(n)\subset\{B\in M(n,\mathbb{R})\mid B+B^\top=0\}.$$

The right side is the space of all skew symmetric $n \times n$-matrices, which is a vector space of dimension $\frac{n(n-1)}{2}$. Since $SO(n)$ is a differentiable manifold of dimension $\frac{n(n-1)}{2}$, its tangent space $T_I SO(n)$ is a vector space of the same dimension. Since both vector spaces have the same dimension, the above inclusion is actually an equality, i.e.,

$$T_I SO(n) = \{B \in M(n, \mathbb{R}) \mid B + B^\top = 0\}.$$