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Riemannian Geometry IV

Solutions, set 3.

Exercise 5.

(a) For $X = \sum f_i \frac{\partial}{\partial x_i}, Y = \sum g_i \frac{\partial}{\partial x_i}$, we have the general formula:

$$[X,Y] = \sum_i \left(\sum_j f_j \frac{\partial g_i}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}.$$

Applying this formula to our case, we obtain

$$\begin{aligned}
[X,Y] = & \left(-x_3 \frac{\partial(2x_3 - x_2)}{\partial x_2} + x_2 \frac{\partial(2x_3 - x_2)}{\partial x_3} \right) \frac{\partial}{\partial x_1} \\
& + \left(-2x_1 \frac{\partial x_3}{\partial x_3} \right) \frac{\partial}{\partial x_2} + \left(-x_1 \frac{\partial u_2}{\partial u_2} \right) \frac{\partial}{\partial x_3} \\
& = (x_3 + 2x_2) \frac{\partial}{\partial x_1} - 2x_1 \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} = Z.
\end{aligned}$$

(b) Identifying $\frac{\partial}{\partial x_i}$ with the standard basis vector $e_i \in \mathbb{R}^3$ yields

$$X(x_1,x_2,x_3) = (2x_3 - x_2, x_1, -2x_1)^\top, \quad Y(x_1,x_2,x_3) = (0, x_3, -x_2)^\top.$$

In order to show that the restrictions of X, Y to S^2 are vector fields on S^2, we have to show that they are tangent to the sphere. The tangent space $T_x S^2$ is given by all vectors $z \in \mathbb{R}^3$ such that $\langle x, z \rangle = 0$. So we check:

$$\begin{aligned}
\langle (x_1,x_2,x_3), X(x_1,x_2,x_3) \rangle &= x_1(2x_3 - x_2) + x_2 x_1 - 2x_3 x_1 = 0, \\
\langle (x_1,x_2,x_3), Y(x_1,x_2,x_3) \rangle &= x_2 x_3 - x_3 x_2 = 0.
\end{aligned}$$

(c) Similarly as in (b), we have to check that $\langle x, Z(x) \rangle = 0$. Using $Z(x_1,x_2,x_3) = (x_3 + 2x_2, -2x_1, -x_1)^\top$, we obtain

$$\langle (x_1,x_2,x_3), Z(x_1,x_2,x_3) \rangle = x_1(x_3 + 2x_2) - 2x_2 x_1 - x_3 x_1 = 0.$$

Exercise 6.

(a) We have

$$[X,Y]f = XYf - YXf = -(YXf - XYf) = -[Y,X]f$$

for all $f \in C^\infty(M)$. This implies that $[X,Y] = -[Y,X]$.

(b) Note that $Z(ag) = aZ(g)$ for constants $a \in \mathbb{R}$ since

$$\frac{\partial}{\partial x_i}(ag) = a\frac{\partial g}{\partial x_i},$$

and the same holds for linear combinations of these basis vector fields. So we have

$$\begin{aligned}
&[aX+bY,Z]f = (aX+bY)Zf - Z(aX+bY)f \\
&= aXZf + bYZf - aZXf - bZYf = a(XZf - ZXf) + b(YZf - ZYf) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad = a[X,Z]f + b[Y,Z]f.
\end{aligned}$$

(c) Using (a), it is enough to show that

$$[[X,Y],Z] = [X,[Y,Z]] + [Y,[Z,X]].$$

The left side, applied to a function $f \in C^\infty(M)$ is

$$[[X,Y],Z]f = [X,Y]Zf - Z[X,Y]f = XYZf - YXZf - ZXYf + ZYXf.$$

The right side, applied to the same function is

$$\begin{aligned}
&[X,[Y,Z]]f + [Y,[Z,X]]f = \\
&XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf - ZXYf + XZYf \\
&\qquad\qquad\qquad\qquad = XYZf + ZYXf - YXZf - ZXYf,
\end{aligned}$$

which is notably the same. This proves Jacobi's identity.