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Riemannian Geometry IV

Solutions, set 4.

Exercise 7.

(a) Let $p = (X, Y, Z) \in \mathbb{W}^2$. Then

$$L_p = \{(X, Y, Z) + t(X, Y, Z + 1) \mid t \in \mathbb{R}\}$$

and

$$L_p \cap \mathbb{B}^2 = (X, Y, Z) - \frac{Z}{Z + 1}(X, Y, Z + 1) = (\frac{X}{Z + 1}, \frac{Y}{Z + 1}, 0).$$

We conclude that

$$f(X, Y, Z) = \left(\frac{X}{Z + 1}, \frac{Y}{Z + 1}, 0\right) \in \mathbb{B}^2.$$

Conversely, let $f^{-1}(x, y, 0) = (X, Y, Z)$ with $X^2+Y^2 = Z^2-1$. Since $x = \frac{X}{Z+1}$, $y = \frac{Y}{Z+1}$, we conclude that

$$x^2 + y^2 = \frac{X^2 + Y^2}{(Z + 1)^2} = \frac{Z^2 - 1}{(Z + 1)^2} = \frac{Z - 1}{Z + 1},$$

so $Z = \frac{1+x^2+y^2}{1-x^2-y^2}$ and $Z + 1 = \frac{2}{1-x^2-y^2}$. We conclude that

$$\begin{aligned} f^{-1}(x, y, 0) = (X, Y, Z) &= ((Z + 1)x, (Z + 1)y, Z) \\ &= \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2}\right). \end{aligned}$$

(b) We have

$$\psi^{-1}(y_1, y_2) = f \circ \varphi^{-1}(y_1, y_2) = \left(\frac{\sinh y_2}{1 + \cosh y_2} \cos y_1, \frac{\sinh y_2}{1 + \cosh y_2} \sin y_1, 0\right).$$

(c) Let $p = \varphi^{-1}(x_1, x_2) = (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2)$. Then $f(p) = \psi^{-1}(x_1, x_2)$, so $y_1 = x_1$, $y_2 = x_2$. So

$$f(p) = \left(\frac{\sinh x_2}{1 + \cosh x_2} \cos x_1, \frac{\sinh x_2}{1 + \cosh x_2} \sin x_1, 0 \right).$$

Moreover, we have

$$\begin{aligned}
\left.\frac{\partial}{\partial y_1}\right|_{f(p)} &= \frac{1}{1 + \cosh x_2} \left(-\sinh x_2 \sin x_1, \sinh x_2 \cos x_1, 0\right), \\
\left.\frac{\partial}{\partial y_2}\right|_{f(p)} &= \frac{1}{1 + \cosh x_2} \left(\cos x_1, \sin x_1, 0\right).
\end{aligned}$$

We know from Example 14 (a) that

$$\begin{aligned}
\left\langle \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_1}\right|_p \right\rangle_p &= \sinh^2 x_2, \\
\left\langle \left.\frac{\partial}{\partial x_1}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p \right\rangle_p &= 0, \\
\left\langle \left.\frac{\partial}{\partial x_2}\right|_p, \left.\frac{\partial}{\partial x_2}\right|_p \right\rangle_p &= 1.
\end{aligned}$$

We obviously have

$$\left\langle \left.\frac{\partial}{\partial y_1}\right|_{f(p)}, \left.\frac{\partial}{\partial y_2}\right|_{f(p)} \right\rangle_{f(p)} = 0.$$

Since

$$\|f(p)\|_0^2 = \frac{\sinh^2 x_2}{(1 + \cosh x_2)^2},$$

(where $\|\cdot\|_0$ denotes the standard Euclidean norm,) we conclude that

$$1 - \|f(p)\|_0^2 = \frac{2}{1 + \cosh x_2}.$$

We calculate

$$\begin{aligned}
&\left\langle \left.\frac{\partial}{\partial y_1}\right|_{f(p)}, \left.\frac{\partial}{\partial y_1}\right|_{f(p)} \right\rangle_{f(p)} = \\
&\quad \frac{4}{(1 - \|f(p)\|_0^2)^2} \frac{1}{(1 + \cosh x_2)^2} \|(-\sinh x_2 \sin x_1, \sinh x_2 \cos x_1, 0)\|_0^2 = \\
&\qquad \sinh^2 x_2 \sin^2 x_1 + \sinh^2 x_2 \cos^2 x_1 = \sinh^2 x_2.
\end{aligned}$$

Similarly, we obtain

$$\left\langle \frac{\partial}{\partial y_2}\Big|_{f(p)}, \frac{\partial}{\partial y_2}\Big|_{f(p)} \right\rangle_{f(p)} =$$
$$\frac{4}{(1-\|f(p)\|_0^2)^2}\frac{1}{(1+\cosh x_2)^2}\,\|(\cos x_1, \sin x_1, 0)\|_0^2 =$$
$$\cos^2 x_1 + \sin^2 x_1 = 1.$$

Exercise 8. One first checks that $f_{A^{-1}} \circ f_A = \mathrm{id}_{\mathbb{H}^2}$. Next, we calculate

$$\begin{aligned} Df_A(z)(v) &= \frac{d}{dt}\Big|_{t=0} f_A(z+tv) = \frac{d}{dt}\Big|_{t=0} \frac{a(z+tv)+b}{c(z+tv)+d} \\ &= \frac{av(cz+d) = (az+b)cv}{(cz+d)^2} = \frac{(ad-bc)v}{(cz+d)^2} = \frac{1}{(cz+d)^2}v, \end{aligned}$$

where we used in the last step that $\det A = ad - bc = 1$. The imaginary part of $f_A(z)$ is related to the imaginary part of z in the following way:

$$\begin{aligned} \mathrm{Im}(f_A(z)) &= \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\right) \\ &= \frac{1}{|cz+d|^2}\mathrm{Im}(adz + bc\bar{z} + ac|z|^2 + bd) = \frac{(ad-bc)\mathrm{Im}(z)}{|cz+d|} = \frac{\mathrm{Im}(z)}{|cz+d|^2}. \end{aligned}$$

We conclude that

$$\begin{aligned} g_{f_A(z)}\left(Df_A(z)(v), Df_A(z)(v)\right) &= \frac{1}{\mathrm{Im}(f_A(z))^2}\frac{1}{|cz+d|^4}\langle v, v\rangle \\ &= \frac{|cz+d|^4}{\mathrm{Im}(z)}\frac{1}{|cz+d|^4}\langle v, v\rangle = \frac{1}{\mathrm{Im}(z)}\langle v, v\rangle = g_z(v,v), \end{aligned}$$

so f_A is an isometry.