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Riemannian Geometry IV

Solutions, set 5.

Exercise 9. T^2 is the rotation of a circle in the (X,Z)-plane with centre $(R,0,0)$ and radius $r>0$ around the vertical Z-axis.

We first note that

$$\begin{aligned}\frac{\partial}{\partial x_1}|_{\varphi^{-1}(x_1,x_2)} &= (-r\sin x_1\cos x_2, -r\sin x_1\sin x_2, r\cos x_1),\\ \frac{\partial}{\partial x_2}|_{\varphi^{-1}(x_1,x_2)} &= (-(R+r\cos x_1)\sin x_2, (R+r\cos x_1)\cos x_2, 0).\end{aligned}$$

This implies that

$$\big(g_{ij}(\varphi^{-1}(x_1,x_2))\big) = \begin{pmatrix} r^2 & 0 \\ 0 & (R+r\cos x_1)^2 \end{pmatrix}.$$

So $\sqrt{\det(g_{ij}(\varphi^{-1}(x_1,x)2)))} = r(R+r\cos x_1)$. Since $\varphi^{-1}(V)=U\subset T^2$ covers all of T^2 expect a set of measure zero, we obtain

$$\begin{aligned}\mathrm{vol}(T^2)=\mathrm{vol}(U) &= \int_V r(R+r\cos x_1)dx_1dx_2\\ &= \int_0^{2\pi}\int_0^{2\pi} r(R+r\cos x_1)dx_1dx_2 = 4\pi^2 rR.\end{aligned}$$

Exercise 10. The normal vector is given by

$$N(x_1,x_2) = \frac{\frac{\partial}{\partial x_1}\times\frac{\partial}{\partial x_2}}{\|\frac{\partial}{\partial x_1}\times\frac{\partial}{\partial x_2}\|} = (\cos x_1\cos x_2, \cos x_1\sin x_2, \sin x_1).$$

This implies that

$$\begin{aligned}\frac{\partial N}{\partial x_1}(x_1,x_2) &= (-\sin x_1\cos x_2, -\sin x_1\sin x_2, \cos x_1),\\ \frac{\partial N}{\partial x_2}(x_1,x_2) &= (-\cos x_1\sin x_2, \cos x_1\cos x_2, 0),\end{aligned}$$

and

$$
\begin{aligned}
L &= -\langle \frac{\partial N}{\partial x_1}(x_1,x_2), \frac{\partial}{\partial x_1}|_{\varphi^{-1}(x_1,x_2)} \rangle = r, \\
M &= -\langle \frac{\partial N}{\partial x_1}(x_1,x_2), \frac{\partial}{\partial x_2}|_{\varphi^{-1}(x_1,x_2)} \rangle = 0, \\
N &= -\langle \frac{\partial N}{\partial x_2}(x_1,x_2), \frac{\partial}{\partial x_2}|_{\varphi^{-1}(x_1,x_2)} \rangle = (R + r\cos x_1)\cos x_1.
\end{aligned}
$$

Using $E = g_{11}(\varphi^{-1}(x_1,x_2)) = r^2$, $F = g_{12}(\varphi^{-1}(x_1,x_2)) = 0$ and $G = g_{22}(\varphi^{-1}(x_1,x_2)) = (R + r\cos x_1)^2$, we conclude that

$$
K(\varphi^{-1}(x_1,x_2)) = \frac{LN - M^2}{EG - F^2} = \frac{r(R + r\cos x_1)\cos x_1}{r^2(R + r\cos x_1)^2} = \frac{\cos x_1}{r(R + r\cos x_1)}.
$$

$x_1 = \pi/2$ or $x_1 = 3\pi/2$ describe points of the torus intersected with the planes $Z = r$ and $Z = -r$. Note that T^2 lies between these two planes and touches each plane in a circle of radius R. Obviously, one of the principal curvatures at these points is equal to zero while the other is equal to $r > 0$, so the Gaussian curvature vanishes. $x_1 = \pi$ describes the points at the inner circle of the torus, i.e., the horizontal circle in the (X,Y)-plane with radius $R - r > 0$. These points are obvioulsy saddle points and the two principal curvatures have different signs. So the Gaussian curvature is negative at these points.

Next, we calculate

$$
\begin{aligned}
\int_{T^2} K \, d\mathrm{vol} &= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos x_1}{r(R + r\cos x_1)} r(R + r\cos x_1) dx_1 dx_2 \\
&= \int_0^{2\pi}\int_0^{2\pi} \cos x_1 \, dx_1 dx_2 = 0.
\end{aligned}
$$

The Gauss-Bonnet Theorem tells us for any closed, oriented surface $S \subset \mathbb{R}^3$ we have

$$
\frac{1}{2\pi}\int_S K \, d\mathrm{vol} = \chi(S),
$$

where $\chi(S)$ is the Euler characteristic of the surface S and given by $\chi(S) = 2 - 2g$, where g is the genus of the surface. Since the genus of the torus is equal to one, we conclude that $\chi(T^2) = 0$, justifying the above calculated result.

Exercise 11. The formula for the hyperbolic volume yields

$$\begin{aligned}
\mathrm{vol}(\Delta P_1P_2P_3) &= \int_{x_1}^{x_2}\int_{\sqrt{r^2-(x-x_0)^2}}^{\infty}\frac{dy}{y^2}dx = \int_{x_1}^{x_2}\frac{1}{y}\Big|_{\infty}^{\sqrt{r^2-(x-x_0)^2}}dx\\
&= \int_{x_1}^{x_2}\frac{1}{\sqrt{r^2-(x-x_0)^2}}dx = \frac{1}{r}\int_{x_1}^{x_2}\frac{1}{\sqrt{1-\left(\frac{x-x_0}{r}\right)^2}}dx\\
&= \frac{1}{r}\int_{x_1-x_0}^{x_2-x_0}\frac{1}{\sqrt{1-\left(\frac{x}{r}\right)^2}}dx = \int_{\frac{x_1-x_0}{r}}^{\frac{x_2-x_0}{r}}\frac{dy}{\sqrt{1-y^2}}\\
&= \arcsin\left(\frac{x_2-x_0}{r}\right) - \arcsin\left(\frac{x_1-x_0}{2}\right) = \arcsin\left(\frac{x_2-x_0}{r}\right) + \arcsin\left(\frac{x_0-x_1}{2}\right).
\end{aligned}$$

The picture implies that $\cos\alpha = \frac{x_0-x_1}{r}$. Since $\alpha \in (0, \pi/2)$, we have

$$\alpha = \arccos\left(\frac{x_0-x_1}{r}\right).$$

Similarly, we conclude that

$$\beta = \arccos\left(\frac{x_2-x_0}{r}\right).$$

Using $\arcsin(x) + \arccos(x) = \pi/2$ we finally obtain the desired formula:

$$\begin{aligned}
\mathrm{vol}(\Delta P_1P_2P_3) &= \arcsin\left(\frac{x_2-x_0}{r}\right) + \arcsin\left(\frac{x_0-x_1}{2}\right)\\
&= \pi - \arccos\left(\frac{x_2-x_0}{r}\right) - \arccos\left(\frac{x_0-x_1}{2}\right) = \pi - \alpha - \beta.
\end{aligned}$$