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Riemannian Geometry IV

Solutions, set 8.

Exercise 15. (a) We have with the notation of Exercise 14: $f(x_1) = \cos x_1$, $g(x_1) = \sin x_1$, so $(f'(x_1))^2 + (g'(x_1))^2 = 1$. This implies that

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1} \right) = \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{(f'(x_1))^2 + (g'(x_1))^2} = 0, \\
\Gamma_{11}^2 &= \frac{1}{2} g^{22} (g_{12,1} + g_{12,1} - g_{11,2}) = 0, \\
\Gamma_{12}^1 &= \frac{1}{2} g^{11} (g_{11,2} + g_{12,1} - g_{12,1}) = 0 = \Gamma_{21}^1, \\
\Gamma_{12}^2 &= \frac{1}{2} g^{22} (g_{12,2} + g_{22,1} - g_{12,2}) = \frac{f'(x_1)}{f(x_1)} = -\tan x_1 = \Gamma_{21}^2, \\
\Gamma_{22}^1 &= \frac{1}{2} g^{11} (g_{12,2} + g_{12,2} - g_{22,1}) = \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2} = \sin x_1 \cos x_1, \\
\Gamma_{22}^2 &= \frac{1}{2} g^{22} (g_{22,2} + g_{22,2} - g_{22,2}) = 0.
\end{aligned}$$

(b) We are interested in the parallel vector field X along the equator, parametrized by $c : (-\pi, \pi) \to S^2$, $c(t) = \varphi^{-1}(0, \pi + t)$ with initial condition $X(0) = \frac{\partial}{\partial x_1}|_{c(0)}$. This implies that

$$\varphi \circ c_1(t) =: (c_{11}(t), c_{12}(t)) = (0, \pi + t),$$

hence

$$(c_{11}'(t), c_{12}'(t)) = (0, 1).$$

The solution reduces to the ordinary differential equation

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = - \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that for $x_1 = 0$ we have

$$- \begin{pmatrix} \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} 0 & -\sin x_1 \cos x_1 \\ \tan x_1 & 0 \end{pmatrix} = 0.$$

So the differential equations are simply $a_1' \equiv 0 \equiv a_2'$ which, together with $a_1(0) = 1, a_2(0) = 0$ implies that

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So we end up with $X(t) = \frac{\partial}{\partial x_1}|_{c(t)}$.

(c) From

$$A^2 = \begin{pmatrix} -ab & 0 \\ 0 & -ab \end{pmatrix}$$

we conclude that

$$\begin{aligned} A^{2k} &= (-1)^k \begin{pmatrix} (ab)^k & 0 \\ 0 & (ab)^k \end{pmatrix}, \\ A^{2k+1} &= (-1)^k \begin{pmatrix} 0 & -a(ab)^k \\ b(ab)^k & 0 \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} \mathrm{Exp}(tA) &= \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} \begin{pmatrix} (\sqrt{ab})^{2k} & 0 \\ 0 & (\sqrt{ab})^{2k} \end{pmatrix} \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} \begin{pmatrix} 9 & -\sqrt{\frac{a}{b}}(\sqrt{ab})^{2k+1} \\ \sqrt{\frac{b}{a}}(\sqrt{ab})^{2k+1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\sqrt{ab}t) & -\sqrt{\frac{a}{b}}\sin(\sqrt{ab}t) \\ \sqrt{\frac{b}{a}}\sin(\sqrt{ab}t) & \cos(\sqrt{ab}t) \end{pmatrix}. \end{aligned}$$

(d) We have $\varphi \circ c_2(t) = (\pi/4, \pi + t) = (c_{21}(t), c_{22}(t))$ and $Y(0) = \frac{\partial}{\partial x_1}|_{c_2(0)}$. This translates into the ordinary differential equation

$$\begin{pmatrix} a_1' \\ a_2' \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad \begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using (c), we conclude that

$$\begin{aligned} \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} &= \mathrm{Exp}(t \begin{pmatrix} 0 & -1/2 \\ 1 & 0 \end{pmatrix}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\frac{t}{\sqrt{2}}) & -\sqrt{\frac{1}{2}}\sin(\frac{t}{\sqrt{2}}) \\ \sqrt{2}\sin(\frac{t}{\sqrt{2}}) & \cos(\frac{t}{\sqrt{2}}) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\frac{t}{\sqrt{2}}) \\ \sqrt{2}\sin(\frac{t}{\sqrt{2}}) \end{pmatrix}. \end{aligned}$$

So the parallel vector field Y along c_2 is given by

$$Y(t) = \cos(\frac{t}{\sqrt{2}})\frac{\partial}{\partial x_1}\Big|_{c_2(t)} + \sin(\frac{t}{\sqrt{2}})\sqrt{2}\frac{\partial}{\partial x_2}\Big|_{c_2(t)}.$$

Note that $\frac{\partial}{\partial x_1}|_{c_2(t)}$ and $\sqrt{2}\frac{\partial}{\partial x_2}|_{c_2(t)}$ form an orthonormal basis of $T_{c_2(t)}S^2$, so $Y(t)$ is a rotating vector field along the parallel c_2, similarly to the calculations for the hyperbolic plane (see Example 26).

Exercise 16. Assume first that there are global vector fields $\tilde{X}, \tilde{Y} : M \to TM$ with $\tilde{X}(c(t)) = X(t)$ and $\tilde{Y}(c(t)) = Y(t)$ for all $t \in [a,b]$. Since the Levi-Civita connection is Riemannian, we conclude that

$$\frac{d}{dt}\Big|_t \langle X, Y\rangle = \frac{d}{dt}\Big|_t \left(\langle \tilde{X}, \tilde{Y}\rangle \circ c\right) = c'(t)\left(\langle \tilde{X}, \tilde{Y}\rangle\right)$$
$$= \langle \nabla_{c'(t)}\tilde{X}, Y(t)\rangle + \langle X(t), \nabla_{c'(t)}\tilde{Y}\rangle = \langle \frac{D}{dt}X(t), Y(t)\rangle + \langle X(t), \frac{D}{dt}Y(t)\rangle = 0,$$

since the vector fields X, Y are parallel along c. But this implies that $t \mapsto \langle X(t), Y(t)\rangle$ is a constant function, i.e. the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is an isometry, since

$$\langle P_c X(a), P_c Y(a)\rangle = \langle X(b), Y(b)\rangle = \langle X(a), Y(a)\rangle.$$

Now assume that X, Y don't have global extensions. Assume that there is a coordinate chart $\varphi : (x_1, \dots, x_n) : U \to V$ with $c([a,b]) \subset U$. Then we can write

$$X(t) = \sum a_j(t)\frac{\partial}{\partial x_j}\Big|_{c(t)}, \quad Y(t) = \sum b_j(t)\frac{\partial}{\partial x_j}\Big|_{c(t)},$$

and we have

$$\frac{d}{dt}\langle X, Y\rangle = \frac{d}{dt}\left(\sum_{j,k} a_j b_k \left(\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle \circ c\right)\right).$$

As previously, the Riemannian property of the Levi-Civita connection yields

$$\frac{d}{dt}\left(\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle \circ c\right) = \langle \nabla_{c'(t)}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big|_{c(t)}\rangle + \langle \frac{\partial}{\partial x_j}\Big|_{c(t)}, \nabla_{c'(t)}\frac{\partial}{\partial x_k}\rangle.$$

This implies that

$$\begin{aligned}
\frac{d}{dt}\langle X, Y\rangle &= \sum_{i,k}(a_j' b_k + a_j b_k')\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle \circ c + \\
&\quad a_j b_k \left(\langle \nabla_{c'(t)} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big|_{c(t)} \rangle + \langle \frac{\partial}{\partial x_j}\Big|_{c(t)}, \nabla_{c'(t)} \frac{\partial}{\partial x_k} \rangle \right) \\
&= \langle \sum_j a_j'(t) \frac{\partial}{\partial x_j}\Big|_{c(t)} + a_j(t) \nabla_{c'(t)} \frac{\partial}{\partial x_j}, \sum_k b_k(t) \frac{\partial}{\partial x_k}\Big|_{c(t)} \rangle + \\
&\quad \langle \sum_j a_j(t) \frac{\partial}{\partial x_j}\Big|_{c(t)}, \sum_k b_k'(t) \frac{\partial}{\partial x_k}\Big|_{c(t)} + b_k(t) \nabla_{c'(t)} \frac{\partial}{\partial x_k} \rangle \\
&= \langle \sum_j \frac{D}{dt}\left(a_j \frac{\partial}{\partial x_j} \circ c \right), Y \rangle + \langle X, \sum_k \frac{D}{dt}\left(a_k \frac{\partial}{\partial x_k} \circ c \right) \\
&= \langle \frac{D}{dt} X, Y \rangle + \langle X, \frac{D}{dt} Y \rangle = 0.
\end{aligned}$$

Finally, if we need k coordinate charts $U_1, \ldots, U_k$ to cover $c([a, b])$, i.e., if we have

$$c([a, b]) \subset \bigcup_{j=1}^{k} U_j$$

with a partition $a < t_1 < t_2 \cdots < t_{k-1} < b$ such that $c(a), c(t_1) \in U_1, c(t_1), c(t_2) \in U_2, \ldots c(t_{k-1}), c(b) \in U_k$, so we conclude with the previous argument that $\frac{d}{dt}\langle X, Y\rangle$ is constant on the segments $[a, t_1], [t_1, t_2], \ldots, [t_{k-1}, b]$, and therefore, constant on all of $[a, b]$.