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Riemannian Geometry IV

Problems, set 10.

Exercise 23.

(a) Show that

$$\mathrm{Exp}\left(t\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\right)=\begin{pmatrix}1&t&t^2/2&t^3/(3!)\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Guess how the answer would be for the Lie group exponential of a $n\times n$-matrix of the same form (i.e., only entries 1 at the first upper diagonal).

(b) Use the fact (you don't need to prove this) that if A, B commute then

$$\mathrm{Exp}(A)\mathrm{Exp}(B)=\mathrm{Exp}(A+B),$$

in order to show that

$$\mathrm{Exp}\left(t\begin{pmatrix}c&1&0&0\\0&c&1&0\\0&0&c&1\\0&0&0&c\end{pmatrix}\right)=e^{tc}\begin{pmatrix}1&t&t^2/2&t^3/(3!)\\0&1&t&t^2/2\\0&0&1&t\\0&0&0&1\end{pmatrix}.$$

Exercise 24. Let $(G,\langle\cdot,\cdot\rangle)$ be a *compact* Lie group with left-invariant metric and let *dvol* denote the corresponding left-invariant volume form. Compactness implies that $vol(G)<\infty$ (you don't need to prove this). Define an inner product $\langle\langle\cdot,\cdot\rangle\rangle_e$ at $e\in G$ by

$$\langle\langle v_1,v_2\rangle\rangle_e := \int_G \langle Ad(g^{-1})v_1, Ad(g^{-1})v_2\rangle_e \, dvol(g),$$

and let $\langle\langle\cdot,\cdot\rangle\rangle_g$ denote the left-invariant extension to a Riemannian metric on G. Show that $\langle\langle\cdot,\cdot\rangle\rangle_g$ is a bi-invariant Riemannian metric on G:

(a) Show first that

$$\langle\langle Ad(h^{-1})v_1, Ad(h^{-1})v_2\rangle\rangle_e = \langle\langle v_1, v_2\rangle\rangle_e$$

for all $h \in G$, by using the fact that left-invariance of $dvol$ implies that

$$\int_G f(L_h(g))\, dvol(g) = \int_G f(g)\, dvol(g).$$

(You may use this fact without proof.)

(b) Use the fact $Ad(h^{-1}) = DL_{h^{-1}}(h)DR_h(e)$ in order to show

$$\langle\langle DR_h(e)v_1, DR_h(e)v_2\rangle\rangle_h = \langle\langle v_1, v_2\rangle\rangle_e \quad \text{for all } h \in G,$$

i.e., the right-invariance of $\langle\langle\cdot,\cdot\rangle\rangle_g$.

Remark: The above averaging procedure is called the *Weyl trick*.

Exercise 25. Let $(G, \langle\cdot,\cdot\rangle)$ be a Lie group with a *bi-invariant* Riemannian metric. Let $\mathfrak{g}$ denote the corresponding Lie algebra of left-invariant vector fields on G. Show for $X, Y \in \mathfrak{g}$ that

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Hint: Use the relation

$$\langle X, \nabla_Z Y\rangle = \frac{1}{2}\left(Z\langle X, Y\rangle + Y\langle X, Z\rangle - X\langle Y, Z\rangle + \langle Z, [X, Y]\rangle + \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle\right)$$

and the fact that the metric is left-invariant to prove that $\langle X, \nabla_Y Y\rangle = \langle Y, [X, Y]\rangle$ for $X, Y, Z \in \mathfrak{g}$. Use also the fact that the bi-invariance of the metric implies that

$$\langle [U, X], V\rangle = -\langle U, [V, X]\rangle$$

for $X, U, V \in \mathfrak{g}$ (see Corollary 4.10) in order to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

Merry Christmas and Happy New Year!!!