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Riemannian Geometry IV

Problems, set 13.

Exercise 29. Let (M, g) be a Riemannian manifold, $p \in M$, $\epsilon > 0$ as in the Gauss-Lemma and $B_\epsilon(p) := \exp_p(B_\epsilon(0_p))$. Let a curve $c : [a, b] \to B_\epsilon(p) \backslash \{p\}$ be given by

$$c(s) = \exp_p r(s)v(s),$$

where $v(s) \in S_pM = \{v \in T_pM \mid \|v\|_p = 1\}$ for all $s \in [a, b]$ (polar coordinates). Show that the length $l(c)$ satisfies

$$l(c) \geq |r(b) - r(a)|,$$

with equality if and only if $s \to v(s)$ is constant and r is monotone increasing or decreasing, i.e., the trace of c coincides with part of a radial geodesic.

Hint: Introduce $F(s, t) := \exp_p(tv(s))$. Then $c(s) = F(s, r(s))$. Use the Gauß-Lemma. This exercise is similar in spirit to Example 19 of the lecture.

Exercise 30. In this exercise we discuss a useful coordinate system, called *geodesic normal coordinates*.

Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \dots, v_n$ be a orthonormal base of T_pM. Then a local coordinate chart of M is given by $\varphi = (x_1, \dots, x_n) : B_\epsilon(p) \to V := \{w \in \mathbb{R}^n \mid |w| < \epsilon\}$ via

$$\varphi^{-1}(x_1, \dots, x_n) = \exp_p(\sum_{i=1}^n x_i v_i).$$

The coordinate functions $x_1, \dots, x_n$ of φ are called geodesic normal coordinates.

(a) Let g_{ij} be the first fundamental form in terms of the above coordinate system φ. Show that at $p \in M$:

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ be arbitrarily and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0,$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point $p \in M$.

Exercise 31. Let (M, g) be a n-dimensional Riemannian manifold and $\pi : TM \to M$ be the footpoint projection. For $v \in T_pM$, let

$$\Psi : T_vTM \to T_pM \times T_pM, \quad X'(0) \mapsto \left((\pi \circ X)'(0), \frac{D}{dt} X(0) \right)$$

be the isomorphism introduced in the lecture (here $X : (-\epsilon, \epsilon) \to TM$ is a curve in the tangent bundle representing a tangent vector of the $2n$-dimensional manifold TM, and $\frac{D}{dt}$ denotes the covariant derivative along the projected curve $\pi \circ X : (-\epsilon, \epsilon) \to M$). $SM := \{v \in TM \mid \|v\| = 1\}$ is a $2n-1$-dimensional submanifold of TM (you do not need to prove this). Show that

$$\Psi(T_vSM) = \{(w_1, w_2) \in T_pM \times T_pM \mid w_2 \perp v \text{ w.r.t } g_p\}.$$