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Riemannian Geometry IV

Problems, set 14.

Exercise 32. Let (M, g) be a connected Riemannian manifold, $d_g : M \times M \to [0, \infty)$ be the induced distance function, and Φ_t be the associated geodesic flow.

(a) Assume that (M, d_g) is compact. Show that (M, d_g) is complete.

(b) Assume that (M, d_g) is complete. Conclude that the geodesic flow is defined on all of TM.

Exercise 33. Let (M, g) be a Riemannian manifold and R its curvature tensor. For (b) and (c) below you may also use the results of Proposition 6.2.

(a) Show that
$$R(fX, Y)Z = fR(X, Y)Z,$$
for $f \in C^\infty(M)$ and X, Y, Z vector fields on M.

(b) Show that
$$R(X, fY)Z = fR(X, Y)Z,$$
for $f \in C^\infty(M)$ and X, Y, Z vector fields on M.

(c) Show that
$$\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle,$$
for $f \in C^\infty(M)$ and X, Y, Z, W vector fields on M.

(d) Conclude from (a),(b),(c) that
$$R(fX, gY)hZ = fghR(X, Y)Z,$$
for $f, g, h \in C^\infty(M)$ and X, Y, Z vector fields on M.

Exercise 34. Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,$$

for X, Y, Z vector fields on M, by reducing the equation to Jacobi's identity.

Exercise 35. Let (M, g) be a Riemannian manifold and $v_1, \ldots, v_n \in T_pM$ be an orthonormal basis. We know from Exercise 30 for the geodesic normal coordinates $\varphi : B_\epsilon(p) \to B_\epsilon(0) \subset \mathbb{R}^n$,

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum x_i v_i)$$

that $\frac{\partial}{\partial x_i}|_p = v_i$ and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. Define an orthonormal frame $E_1, \ldots, E_n : B_\epsilon(p) \to TM$ by Gram-Schmidt orthonormalisation, i.e.,

$$\begin{aligned}
F_1(q) &:= \left.\frac{\partial}{\partial x_1}\right|_q, \qquad E_1(q) := \frac{1}{\|F_1(q)\|} F_1(q), \\
&\vdots \\
F_k(q) &:= \left.\frac{\partial}{\partial x_k}\right|_q - \sum_{j=1}^{k-1} \left\langle \left.\frac{\partial}{\partial x_k}\right|_q, E_j(q) \right\rangle E_j(q), \qquad E_k(q) := \frac{1}{\|F_k(q)\|} F_k(q), \\
&\vdots
\end{aligned}$$

By construction, we have $E_i(p) = v_i$ and $E_1(q), \ldots, E_n(q)$ are orthonormal in T_qM for all $q \in B_\epsilon(p)$. Show that

$$\left(\nabla_{E_i} E_j\right)(p) = 0$$

for all $i, j \in \{1, \ldots, n\}$.

Hint: Prove first by induction over k that

$$\begin{aligned}
\left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) &= 0, \\
\nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) &= 0, \\
\left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) &= 0,
\end{aligned}$$

for all $i \in \{1, \ldots, n\}$.