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Riemannian Geometry IV

Problems, set 16.

Do **Exercise 40** as homework for this week. The cumulative homework over the coming weeks will be collected and marked in a few weeks time.

Exercise 39. Let (M, g) be a Riemannian manifold and $p \in M$ be fixed. Assume there exists a constant C such that $K(\Sigma) = C$ for all 2-dimensional subspaces $\Sigma \subset T_pM$. The goal of this exercise is to show that, for all $v_1, v_2, v_3, v_4 \in T_pM$,

$$\langle R(v_1, v_2)v_3, v_4\rangle = C\left(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\right). \tag{1}$$

This goal can be established via the following steps. Let us, for simplicity, introduce the notions

$$\begin{aligned} (v_1, v_2, v_3, v_4) &:= \langle R(v_1, v_2)v_3, v_4\rangle, \\ (v_1, v_2, v_3, v_4)' &:= C\left(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\right). \end{aligned}$$

(a) Check that $(v_1, v_2, v_3, v_4)'$ has the same symmetries as (v_1, v_2, v_3, v_4), namely

  (i) $(v_1, v_2, v_3, v_4)' = -(v_2, v_1, v_3, v_4)'$

  (ii) $(v_1, v_2, v_3, v_4)' + (v_2, v_3, v_1, v_4)' + (v_3, v_1, v_2, v_4)' = 0$

  (iii) $(v_1, v_2, v_3, v_4)' = -(v_1, v_2, v_4, v_3)'$

  (iv) $(v_1, v_2, v_3, v_4)' = (v_3, v_4, v_1, v_2)'$

(b) Show that

$$(v_1, v_2, v_3, v_1) = (v_1, v_2, v_3, v_1)',$$

by starting with the expression $(v_1, v_2 + v_3, v_2 + v_3, v_1)$ and using the fact that $K(\Sigma) = C$ for all 2-dimensional subspaces $\Sigma \subset T_pM$.

(c) Conclude from (b) that

$$(v_1, v_2, v_3, v_4) + (v_4, v_2, v_3, v_1) = (v_1, v_2, v_3, v_4)' + (v_4, v_2, v_3, v_1)'.$$

(d) Derive from (c) that

$$(v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)' = (v_3, v_1, v_2, v_4) - (v_3, v_1, v_2, v_4)',$$

which means that the expression $(v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)'$ is invariant under cyclic permutation of the first three entries.

(e) Using Bianchi's first identity for the Riemannian curvature tensor and property (ii) of $(\cdot, \cdot, \cdot, \cdot)'$, show that

$$(v_1, v_2, v_3, v_4) - (v_1, v_2, v_3, v_4)' = 0,$$

which implies (**??**).

Exercise 40. Show that a manifold with constant sectional curvature is an Einstein manifold. **Hint:** Use the result of Exercise 39.

Exercise 41. Let (M, g) be a Riemannian manifold. For a tensor T let ∇T denote its covariant derivative, as defined in Exercise 19. T is called a parallel tensor, if we have $\nabla T = 0$.

(a) Assume that $T_1, T_2 : \mathcal{X} \times \mathcal{X} \to C^\infty(M)$ are parallel tensors. Show that then the tensor $T : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2) T_2(X_3, X_4),$$

is also parallel.

(b) Use (a) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle.$$

(c) Use Exercise 39 and (b) to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y) Z, W \rangle.$$