Dr. Norbert Peyerimhoff, Durham University 8/11/2010

Riemannian Geometry IV

Problems, set 5.

Exercise 10. Let $\mathbb{W}^2 = \{x \in \mathbb{R}^3 \mid q(x,x) = -1, x_3 > 0\}$ with $q(x,y) = x_1y_1 + x_2y_2 - x_3y_3$ be the hyperboloid model of the hyperbolic plane and $\mathbb{B}^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 < 1, x_3 = 0\}$ be the embedding of the Poincaré ball model into $\mathbb{R}^3$. For every point $p \in \mathbb{W}^2$ let L_p denote the Euclidean straight line through p and $(0,0,-1)$. Let $f : \mathbb{W}^2 \to \mathbb{B}^2$ be the stereographic projection, defined as follows: $f(p) = L_p \cap \mathbb{B}^2$.

(a) Calculate explicitely the maps $f(X,Y,Z)$ for $(X,Y,Z) \in \mathbb{W}^2$ and $f^{-1}(x,y,0)$ for $(x,y,0) \in \mathbb{B}^2$.

(b) A coordinate chart $\varphi : U \to V$ of $\mathbb{W}^2$ is given by

$$\varphi^{-1}(x_1,x_2) = (\cos x_1 \sinh x_2, \sin x_1 \sinh x_2, \cosh x_2),$$

for $(x_1,x_2) \in V = (0,2\pi) \times (0,\infty)$. Let $\psi = \varphi \circ f^{-1}$ be a coordinate chart of $\mathbb{B}^2$ with coordinate functions y_1, y_2. Calculate ψ^{-1} explicitely.

(c) Show that

$$\langle \frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p \rangle_p = \langle \frac{\partial}{\partial y_i}\Big|_{f(p)}, \frac{\partial}{\partial y_j}\Big|_{f(p)} \rangle_{f(p)} \quad \text{for all } p \in U.$$

Using Lemma 2.4, this shows that f is an isometry.

Additional remark: To be precise, one would have to choose two coordinate charts of the above type with $V_1 = (0,2\pi) \times (0,\infty)$ and $V_2 = (-\pi,\pi) \times (0,\infty)$ and an extra consideration for the linear map $Df(0,0,1) : T_{(0,0,1)}\mathbb{W}^2 \to T_0\mathbb{B}^2$ to cover all of the hyperbolic plane and to fully prove that f is an isometry.

Exercise 11. Let

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

be the upper half plane model of the hyperbolic plane. The group $SL(2,\mathbb{R}) = \{A \in M(2,\mathbb{R}) \mid \det A = 1\}$ acts on $\mathbb{H}^2$ in the following way: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Then

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2, \quad f_A(z) = \frac{az+b}{cz+d}.$$

(a) Show that the maps $f_A : \mathbb{H}^2 \to \mathbb{H}^2$ are isometries of the upper half plane model. **Hint:** First prove

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

(b) Show that $\{f_A(i) \mid A \in SL(2,\mathbb{R})\} = \mathbb{H}^2$. **Hint:** Calculate $f_A(z)$ for $A = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ and for $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

(c) Show that $\{A \in SL(2,\mathbb{R}) \mid f_A(i) = i\} = SO(2)$.

Exercise 12. An almost global coordinate chart of the torus of revolution $T^2 \subset \mathbb{R}^3$ is given by $\varphi : U \to V = (0,2\pi) \times (0,2\pi) \subset \mathbb{R}^2$,

$$\varphi^{-1}(x_1,x_2) = ((R + r\cos x_1)\cos x_2, (R + r\cos x_1)\sin x_2, r\sin x_1),$$

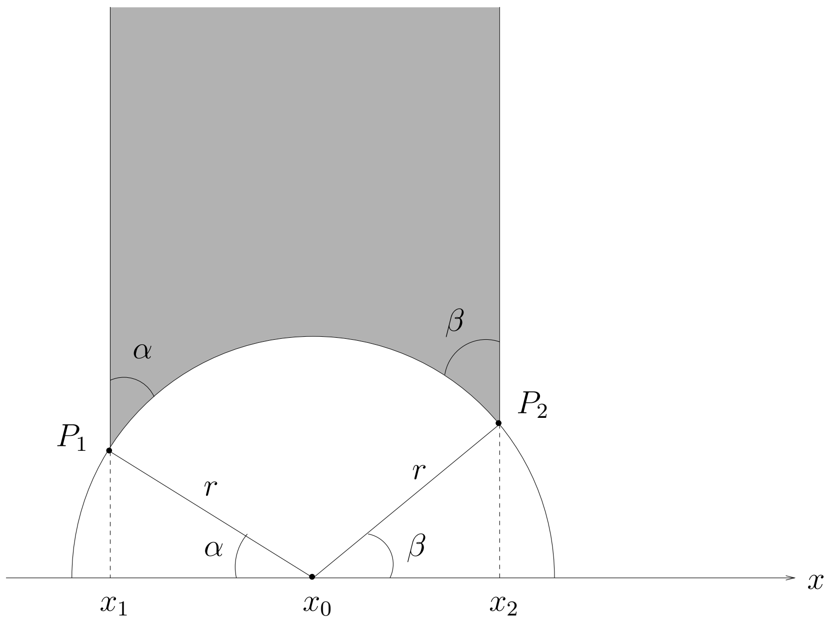
with $r > 0$ and $R > r$. Explain the precise geometric shape which is rotated around the vertical axis in $\mathbb{R}^3$ in order to obtain T^2. Show that

$$\mathrm{vol}(T^2) = (2\pi r)(2\pi R) = 4\pi^2 rR.$$

Exercise 13. We derive the formula

$$\mathrm{vol}(\Delta P_1P_2P_3) = \pi - \alpha - \beta$$

for a triangle $\Delta P_1P_2P_3$ in the upper half space model $\mathbb{H}^2$ with an ideal vertex P_3 and interior angles α, β. Look at the following picture and denote the "ideal" point at infinity by P_3.



Show that

$$\text{vol}(\Delta P_1 P_2 P_3) = \arcsin(\frac{x_0 - x_1}{r}) + \arcsin(\frac{x_2 - x_0}{r}).$$

(You may use $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.) Conclude from this with $\arcsin(x) + \arccos(x) = \pi/2$ that

$$\text{vol}(\Delta P_1 P_2 P_3) = \pi - \alpha - \beta.$$