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Riemannian Geometry IV

Problems, set 9.

Exercise 21. An almost global coordinate chart of the sphere $S^2 \subset \mathbb{R}^3$ is given by $\varphi : U \to V = (-\pi/2, \pi/2) \times (0, 2\pi)$, $U \subset S^2$,

$$\varphi^{-1}(x_1, x_2) = (\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1).$$

We want to determine the parallel vector fields along certain parallels of the sphere.

(a) Determine the Christoffel symbols with respect to this cordinate system. **Hint:** You may use the Solution of Exercise 20.

(b) Let X be the parallel vector field along $c_1 : (-\pi, \pi) \to S^2$, $c_1(t) = \varphi^{-1}(0, \pi + t)$ with $X(0) = \frac{\partial}{\partial x_1}|_{c_1(0)}$. Calculate X explicitely in terms of the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$.

(c) For $a, b > 0$ let

$$A = \begin{pmatrix} 0 & -a \\ b & 0 \end{pmatrix}.$$

Show that

$$\operatorname{Exp}(tA) = \begin{pmatrix} \cos(\sqrt{ab}t) & -\sqrt{\frac{a}{b}} \sin(\sqrt{ab}t) \\ \sqrt{\frac{b}{a}} \sin(\sqrt{ab}t) & \cos(\sqrt{ab}t) \end{pmatrix}.$$

(d) Let Y be the parallel vector field along $c_2 : (-\pi, \pi) \to S^2$, $c_2(t) = \varphi^{-1}(\pi/4, \pi + t)$ with $Y(0) = \frac{\partial}{\partial x_1}|_{c_2(0)}$. Calculate Y explicitely in terms of the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$.

Exercise 22. Let (M, g) be a Riemannian manifold and $c : [a, b] \to M$ be a differentiable curve. Let $\frac{D}{dt}$ denote the corresponding covariant derivative along the curve c. Show the following: For any two parallel vector fields X, Y along c we have

$$\frac{d}{dt}\langle X, Y\rangle \equiv 0,$$

i.e., the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry.

Hint: I am happy if you try to prove this statement in the particular case that the vector fields X, Y along c have global extensions $\tilde{X}, \tilde{Y} : M \to TM$.

However, be aware that not every vector field along a curve may have a global extension; an extreme example is the case where $c : [a, b] \to M$ is a constant map $c(t) = p$ for all $t \in [a, b]$ and $X(t)$ is varying in T_pM. In this case, you need to write X, Y with respect to a basis of a coordinate system. Therefore, it is convenient to assume that $c([a, b])$ is contained in the domain of a coordinate chart. But this assumption is not a serious restriction, for otherwise one covers $c([a, b])$ with a finite sequence of covering coordinate charts and argues locally (see the solution sheet).