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# Riemannian Geometry IV

**Solutions, set 1.**

**Exercise 1.** Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ and $\{(\widetilde{U}_\beta, \widetilde{\varphi}_\beta)\}_{\beta \in \widetilde{\mathcal{A}}}$ be atlases of $M$ and $N$, respectively. Then an atlas of $M \times N$ is given by $\{(U_\alpha \times \widetilde{U}_\beta, \psi_{\alpha,\beta})\}_{(\alpha,\beta) \in \mathcal{A} \times \widetilde{\mathcal{A}}}$, where

$$\psi_{\alpha,\beta} : U_\alpha \times \widetilde{U}_\beta \to V_\alpha \times \widetilde{V}_\beta \subset \mathbb{R}^{m+n}$$

with

$$\psi_{\alpha,\beta}(x,y) := (\varphi_\alpha(x), \widetilde{\varphi}_\beta(y)).$$

The coordinate changes are

$$\psi_{\gamma,\delta}^{-1} \circ \psi_{\alpha,\beta}(x,y) = (\varphi_\gamma^{-1} \circ \varphi_\alpha(x), \widetilde{\varphi}_\delta^{-1} \circ \widetilde{\varphi}_\beta(y)),$$

which are obviously differentiable.

Finally, we have to check the Hausdorff property: Let $(x,y) \neq (z,w)$. This means that $x \neq z$ or $y \neq w$. Choose open neighbourhoods $U_x, U_z$ of $x, z \in M$ which do not intersect if $x \neq z$. Choose open neighbourhoods $\widetilde{U}_y, \widetilde{U}_w$ of $y, w \in N$ which do not intersect if $y \neq w$. Then $U_x \times \widetilde{U}_y \subset M \times N$ and $U_z \times \widetilde{U}_w \subset M \times N$ are open neighbourhoods of $(x,y)$ and $(z,w)$, respectively, and

$$(U_x \times \widetilde{U}_y) \cap (U_z \times \widetilde{U}_w) = \emptyset.$$

**Exercise 2.** Let

$$f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x_1,x_2,x_3) = \left(\sqrt{x_1^2+x_2^2} - R\right)^2 + x_3^2.$$

We have to show that $r^2$ is a regular value of $f$, then $M = f^{-1}(r^2)$ is a differentiable manifold of dimension $(3-1) = 2$, by Theorem 1.5. We have

$$Df(x_1,x_2,x_3) = 2\left( (\sqrt{x_1^2+x_2^2} - R)\frac{x_1}{\sqrt{x_1^2+x_2^2}}, (\sqrt{x_1^2+x_2^2} - R)\frac{x_2}{\sqrt{x_1^2+x_2^2}}, x_3 \right).$$

Obviously, $Df(x_1, x_2, x_3)$ is surjective if $x_3 \neq 0$. Assume now that $x_3 = 0$. If $(x_1, x_2, 0) \in f^{-1}(r^2)$, then

$$\sqrt{x_1^2 + x_2^2} - R = \pm r,$$

i.e., $\sqrt{x_1^2 + x_2^2} = R \pm r > 0$, since $R > r$. This implies that $x_1 \neq 0$ or $x_2 \neq 0$, which means that either the first of second component of $Df(x_1, x_2, x_3)$ is non-vanishing, i.e., $Df(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$ is surjective.

Finally, we have to construct the diffeomorphism $\Phi$. We map the point $(x, y) \in S^1$ of the first factor of $S^1 \times S^1$ to the point $(R + rx, 0, ry)$ in the $x_1, x_3$-plane, and rotate this with the point $(\tilde{x}, \tilde{y}) \in S^1$ of the second factor of $S^1 \times S^1$ around the $x_3$-axis to obtain $(\tilde{x}(R+rx), \tilde{y}(R+rx), ry)$. Therefore,

$$\Phi : S^1 \times S^1 \to M, \quad \Phi(x, y, \tilde{x}, \tilde{y}) = (\tilde{x}(R + rx), \tilde{y}(R + rx), ry).$$

**Exercise 3.**

(a) Let $g(\lambda) = f(\lambda x_1, \ldots, \lambda x_k) = \lambda^m f(x_1, \ldots, x_k)$. Using the chain rule, we obtain

$$\sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(\lambda x) x_i = m\lambda^{m-1} f(x_1, \ldots, x_k).$$

Choosing $\lambda = 1$, the left side is equal to $\langle \mathrm{grad} f(x), x \rangle$ and the right side is $mf(x)$. This finishes the proof of (a).

(b) Let $f$ be a homogeneous polynomial of degree $m \geq 1$ and $y \neq 0$. Let $x \in f^{-1}(y)$. Then we obtain with (a):

$$\langle \mathrm{grad} f(x), x \rangle = mf(y) \neq 0.$$

This implies that $\mathrm{grad} f(x) \neq 0$, so $Df(x) : \mathbb{R}^k \to \mathbb{R}$ is surjective for all $x \in f^{-1}(y)$. Therefore, $y \neq 0$ is a regular value.

(c) The group $SL(n, \mathbb{R}) \subset M(n, \mathbb{R}) = \mathbb{R}^{n^2}$ ist equal to $f^{-1}(1)$, where $f(A) = \det A$. Now, $f$ is a homogeneous polynomial of degree $n$ in $\mathbb{R}^{n^2}$, so 1 is a regular value of $f$, by (b). Theorem 1.5 implies that $SL(n, \mathbb{R}) = f^{-1}(1)$ is a differentiable manifold of dimension $n^2 - 1$.