Riemannian Geometry IV

Solutions, set 12.

Exercise 28. We have

$$E'(0) = \frac{d}{ds}\Big|_{s=0} \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2 dt = \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\Big|_{s=0} \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle dt =$$
$$\int_a^b \langle \frac{D}{ds}\frac{\partial F}{\partial t}(0,t), c'(t)\rangle dt.$$

Applying the Symmetry Lemma yields

$$E'(0) = \int_a^b \langle \frac{D}{dt}\frac{\partial F}{\partial s}(0,t), c'(t)\rangle dt = \int_a^b \frac{d}{dt}\langle X(t), c'9t)\rangle - \langle X(t), \frac{D}{dt}c'(t)\rangle dt =$$
$$\langle X(b), c'(b)\rangle - \langle X(a), c'(a)\rangle - \int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt.$$

(i) If c is a geodesic, this simplifies to $E'(0) = \langle X(b), c'(b)\rangle - \langle X(a), c'(a)\rangle$.

(ii) If F is a proper variation, this simplifies to $E'(0) = -\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt$.

(iii) If c is a geodesic and F is a proper variation, this simplifies to $E'(0) = 0$.

Assume that c is not a geodesic. Then there exists a $t_0 \in [a,b]$ with $\frac{D}{dt}c'(t_0) \neq 0$. Since the map $t \to \frac{D}{dt}c'(t_0)$ is continuous, we can assume, w.l.o.g, that $t_0 \in (a,b)$. Choose a smooth function $\varphi : [a,b] \to [0,1]$ with $\varphi(a) = \varphi(b) = 0$ and $\varphi(t_0) = 1$ and set $X(t) = \varphi(t)\frac{D}{dt}c'(t)$. Then X is the variational vector field of a proper variation F, and we obtain for its energy functional

$$E'(0) = -\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt = -\int_a^b \varphi(t)\|\frac{D}{dt}c'(t)\| dt < 0.$$

So we have proved

$$c \text{ no geodesic} \quad \Rightarrow \quad E'(0) \neq 0 \text{ for some proper variation},$$

which is equivalent to (iv).

Finally, assume that c minimises energy amongst all curves $\gamma : [a, b] \to M$ connecting p, q. Let F be a proper variation. Then the curves $t \mapsto F(s, t)$ are also curves $[a, b] \to M$ connecting p, q, so their energy is $\geq E(0) = E(c)$. This implies that $E'(0) = 0$. Using (iv), we conclude that c is a geodesic, proving (v).