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Riemannian Geometry IV

Solutions, set 13.

Exercise 29. We have

$$c'(s) = \frac{\partial F}{\partial s}(s, r(s)) + \frac{\partial F}{\partial t}(s, r(s)) r'(s),$$

by the chain rule. Note that $\| \frac{\partial F}{\partial t}(s,t) \| = \| v(s) \| = 1$ (since $t \mapsto F(s,t)$ is a geodesic with initial vector $v(s)$) and $\frac{\partial F}{\partial s}(s,t) \perp \frac{\partial F}{\partial t}(s,t)$, by the Gauß-Lemma. Therefore,

$$\| c'(s) \| = \sqrt{|r'(s)|^2 + \| \frac{\partial F}{\partial s}(s, r(s)) \|^2} \ge |r'(s)|,$$

and we conclude that

$$l(c) = \int_a^b \| c'(s) \| ds \ge \int_a^b |r'(s)| ds \ge | \int_a^b r'(s) ds | = |r(b) - r(a)|,$$

with equality in the first inequality if and only if $\| \frac{\partial F}{\partial s}(s, r(s)) \equiv 0$ and equality in the second inequality if and only if $r' \ge 0$ or $r' \le 0$ on $[a, b]$. Hence: We have equality if and only if r is monotone and $v(s)$ is a constant function $\equiv v$, i.e., $c(s) = \exp_p r(s) v$.

Exercise 30. (a) Note that $\varphi(p) = 0$, so

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{d}{dt}\Big|_{t=0} \varphi^{-1}(0 + t e_i) = \frac{d}{dt}\Big|_{t=0} \exp_p(t v_i) = v_i.$$

This implies that

$$g_{ij}(p) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p = \langle v_i, v_j \rangle_p = \delta_{ij}.$$

(b) We have

$$c(t) = \varphi^{-1}(t w_1, \dots, t w_n) = \exp_p(t \sum_j w_j v_j). \qquad (1)$$

Let $v = \sum_j w_j v_j \in T_pM$. Then (1) shows that c is a geodesic with initial vector v. Let $(c_1, \dots, c_n) = \varphi \circ c$, i.e., $c_j(t) = tw_j$, $c_j'(t) = w_j$ and $c_j''(t) = 0$. Let $\frac{D}{dt}$ denote covariant derivative along c. Since c is a geodesic, we have

$$0 = \frac{D}{dt}c' = \frac{D}{dt}\sum_j c_j'\left(\frac{\partial}{\partial x_j}\circ c\right) = \sum_j w_j \nabla_{c'}\frac{\partial}{\partial x_j}$$
$$= \sum_{i,j} w_i w_j \left(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}\right)\circ c = \sum_k \left(\sum_{i,j} w_i w_j (\Gamma_{ij}^k \circ c)\right)\frac{\partial}{\partial x_k}\circ c.$$

Using the fact that $\frac{\partial}{\partial x_k}$ form a basis, we conclude that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0, \tag{2}$$

for all $k \in \{1, \dots, n\}$.

(c) Evaluating (2) at $t = 0$, we obtain

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(p) = 0 \quad \text{for all } w \in \mathbb{R}^n.$$

The choice $w = e_i + e_j$ yields

$$2\Gamma_{ij}^k(p) = 0,$$

so we conclude that all Christoffel symbols vanish at p. Consequently, we have

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}(p) = 0.$$

Exercise 31. We first show that

$$\Psi(T_vSM) \subset \{(w_1, w_2) \in T_pM \times T_pM \mid w_2 \perp v \text{ w.r.t } g_p\}.$$

The result follows then immediately from dimension considerations, since both vector spaces have dimension $2n-1$.

Let $X : (-\epsilon, \epsilon) \to SM$ be a curve with $X(0) = v \in S_pM$, representing a tangent vector $X'(0) \in T_vSM$. Let $c = \pi \circ X : (-\epsilon, \epsilon) \to M$ be the corresponding projected curve. Let $\frac{D}{dt}$ denote the covariant derivative along

c. Then $X \in \mathcal{X}_c(M)$ and we have, using the Riemannian property of the Levi-Civita connection,

$$0 = \frac{d}{dt}|_{t=0} \|X(t)\|^2 = 2\, g_{c(t)} \left(\frac{D}{dt} X(t), X(t) \right).$$

Evaluating at $t = 0$ yields

$$0 = 2\, g_p \left(\frac{D}{dt} X(0), v \right),$$

which implies that

$$\Psi(X'(0)) = \left(w_1 = c'(0), w_2 = \frac{D}{dt} X(0) \right)$$

with $g_p(w_2, v) = 0$, i.e., $w_2 \perp v$.