Riemannian Geometry IV

Solutions, set 15.

Exercise 36. Homework! Solution will be provided later!

Exercise 37. (a) We have

$$\begin{aligned}\frac{\partial}{\partial x_1} &= (-r\sin x_1\cos x_2, -r\sin x_1\sin x_2, r\cos x_1),\\ \frac{\partial}{\partial x_2} &= (-r\cos x_1\sin x_2, r\cos x_1\cos x_2, 0).\end{aligned}$$

This implies that

$$(g_{ij}) = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\cos^2 x_1 \end{pmatrix}, \qquad (g^{ij}) = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\cos^2 x_1} \end{pmatrix}.$$

The Christoffel symbols are given by

$$\begin{aligned}\Gamma_{11}^1 &= \frac{1}{2}g^{11}(g_{11,1}+g_{11,1}-g_{11,1}) = 0,\\ \Gamma_{11}^2 &= \frac{1}{2}g^{22}(g_{12,1}+g_{12,1}-g_{11,2}) = 0,\\ \Gamma_{12}^1 = \Gamma_{21}^1 &= \frac{1}{2}g^{11}(g_{11,2}+g_{21,1}-g_{12,1}) = 0,\\ \Gamma_{12}^2 = \Gamma_{21}^2 &= \frac{1}{2}g^{22}(g_{12,2}+g_{22,1}-g_{12,2}) = \frac{1}{2r^2\cos^2 x_1}(-2r\cos x_1\sin x_1) = -\tan x_1,\\ \Gamma_{22}^1 &= \frac{1}{2}g^{11}(g_{21,2}+g_{21,2}-g_{22,1}) = \frac{1}{2r^2}(2r^2\cos x_1\sin x_1) = \sin x_1\cos x_1,\\ \Gamma_{22}^2 &= \frac{1}{2}g^{22}(g_{22,2}+g_{22,2}-g_{22,2}) = 0.\end{aligned}$$

This implies that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1} &= \Gamma_{11}^1\frac{\partial}{\partial x_1}+\Gamma_{11}^2\frac{\partial}{\partial x_2} = 0,\\ \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} = \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_1} &= \Gamma_{12}^1\frac{\partial}{\partial x_1}+\Gamma_{12}^2\frac{\partial}{\partial x_2} = -\tan x_1\frac{\partial}{\partial x_2},\\ \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2} &= \Gamma_{22}^1\frac{\partial}{\partial x_1}+\Gamma_{22}^2\frac{\partial}{\partial x_2} = \sin x_1\cos x_1\frac{\partial}{\partial x_1}.\end{aligned}$$

(b) We have

$$
\begin{aligned}
R(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})\frac{\partial}{\partial x_2} &= \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2} - \nabla_{\frac{\partial}{\partial x_2}}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} - \nabla_{[\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}]}\frac{\partial}{\partial x_2} \\
&= \nabla_{\frac{\partial}{\partial x_1}}\left(\cos x_1 \sin x_1 \frac{\partial}{\partial x_1}\right) - \nabla_{\frac{\partial}{\partial x_2}}\left(-\tan x_1 \frac{\partial}{\partial x_2}\right) - \nabla_0 \frac{\partial}{\partial x_2} \\
&= (\cos^2 x_1 - \sin^2 x_1)\frac{\partial}{\partial x_1} + \tan x_1 \sin x_1 \cos x_1 \frac{\partial}{\partial x_1} = \cos^2 x_1 \frac{\partial}{\partial x_1}.
\end{aligned}
$$

(c) For a surface M we have, for every basis v, w of T_pM:

$$
K(p) = \text{Gaussian curvature at } p = K(T_pM) = \frac{\langle R(v,w)w, v\rangle}{\|v\|^2\,\|w\|^2 - \langle v,w\rangle^2},
$$

i.e., the sectional curvature $K(T_pM)$ coincides with the Gaussian curvature $K(p)$ at p. We conclude that

$$
K = \frac{\langle R(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\rangle}{\|\frac{\partial}{\partial x_1}\|^2\cdot\|\frac{\partial}{\partial x_2}\|^2 - \langle\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\rangle^2} = \frac{\cos^2 x_1\,\|\frac{\partial}{\partial x_1}\|^2}{\|\frac{\partial}{\partial x_1}\|^2\, r^2 \cos^2 x_1} = \frac{1}{r^2}.
$$

Exercise 38. Note that $E_{rs}(p) = \nabla_{e_r}E_s - \nabla_{e_s}E_r = 0$. We obtain first

$$
\begin{aligned}
\nabla R(e_i,e_j,e_k,e_l,e_m) &= e_m(\langle R(E_i,E_j)E_k,E_l\rangle) = e_m(\langle R(E_k,E_l)E_i,E_j\rangle) \\
&= \langle \nabla_{e_m}\nabla_{E_k}\nabla_{E_l}E_i - \nabla_{e_m}\nabla_{E_l}\nabla_{E_k}E_i - \nabla_{e_m}\nabla_{E_{kl}}E_i, e_j\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\nabla R(e_i,e_j,e_k,e_l,e_m) + \nabla R(e_i,e_j,e_l,e_m,e_k) + \nabla R(e_i,e_j,e_m,e_k,e_l) \\
&\quad = \langle \nabla_{e_m}\nabla_{E_k}\nabla_{E_l}E_i + \nabla_{e_k}\nabla_{E_l}\nabla_{E_m}E_i + \nabla_{e_l}\nabla_{E_m}\nabla_{E_k}E_i \\
&\qquad - \nabla_{e_m}\nabla_{E_l}\nabla_{E_k}E_i - \nabla_{e_l}\nabla_{E_k}\nabla_{E_m}E_i - \nabla_{e_k}\nabla_{E_m}\nabla_{E_l}E_i \\
&\qquad - \nabla_{e_m}\nabla_{E_{kl}}E_i - \nabla_{e_k}\nabla_{E_{lm}}E_i - \nabla_{e_l}\nabla_{E_{mk}}E_i, e_j\rangle \\
&\quad = \langle R(e_m,e_k,\nabla_{e_l}E_i) + \nabla_{E_{mk}(p)}\nabla_{E_l}E_i - \nabla_{e_l}\nabla_{E_{mk}}E_i \\
&\qquad + R(e_k,e_l,\nabla_{e_m}E_i) + \nabla_{E_{kl}(p)}\nabla_{E_m}E_i - \nabla_{e_m}\nabla_{E_{kl}}E_i \\
&\qquad + R(e_l,e_m,\nabla_{e_k}E_i) + \nabla_{E_{lm}(p)}\nabla_{E_k}E_i - \nabla_{e_k}\nabla_{E_{lm}}E_i, e_j\rangle.
\end{aligned}
$$

Using $\nabla_{e_r} E_s = 0$, all above curvature terms vanish and this result simplifies to

$$\begin{aligned}\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) \\ = \langle R(E_{mk}(p), e_l, e_i) + \nabla_{[E_{mk}, E_l]} E_i + R(E_{kl}(p), e_m, e_i) + \nabla_{[E_{kl}, E_m]} E_i \\ + R(E_{lm}(p), e_k, e_i) + \nabla_{[E_{lm}, E_k} E_i, e_j\rangle.\end{aligned}$$

Using $E_{rs}(p) = 0$, this simplifies further to

$$\begin{aligned}\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) \\ = \langle \nabla_{[E_{mk}, E_l] + [E_{kl}, E_m] + [E_{lm}, E_k]} E_i, e_j\rangle.\end{aligned}$$

Jacobi's identity tell us that $[E_{mk}, E_l] + [E_{kl}, E_m] + [E_{lm}, E_k] = 0$, and therefore we obtain

$$\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) = 0.$$

Since this holds on any choice of basis vectors in every slot, we obtain the same result for any choice of arbitrary tangent vectors in T_pM in each slot, by linearity.