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Riemannian Geometry IV

Solutions, set 17.

Exercise 42. We know from Exercise 41 that the tensor R' is parallel, i.e., $\nabla R' = 0$. We conclude from Exercise 39 that $R = fR'$, and therefore

$$\nabla R(X,Y,Z,W,U) = (Uf)R'(X,Y,Z,W).$$

The Second Bianchi Identity tells us that

$$\nabla R(X,Y,Z,W,T) + \nabla R(X,Y,W,T,Z) + \nabla R(X,Y,T,Z,W) = 0,$$

which yields, using the definition of R':

$$\begin{aligned} 0 = (Tf)(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) \\ + (Zf)(\langle X,T\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,T\rangle) \\ + (Wf)(\langle X,Z\rangle\langle Y,T\rangle - \langle X,T\rangle\langle Y,Z\rangle). \end{aligned}$$

Using the relations $\langle Z(p),W(p)\rangle = \langle Z(p),Y(p)\rangle = \langle Y(p),W(p)\rangle = 0$, $\|Y(p)\| = 1$ and $T = Y$, we conclude that, at p

$$\begin{aligned} 0 = (Tf)(p)(\langle X(p),W(p)\rangle\cdot 0 - \langle X(p),Z(p)\rangle\cdot 0) \\ + (Zf)(p)(\langle X(p),T(p)\rangle\cdot 0 - \langle X(p),W(p)\rangle\cdot 1) \\ + (Wf)(p)(\langle X(p),Z(p)\rangle\cdot 1 - \langle X(p),T(p)\rangle\cdot 0) \\ = \langle (Wf)(p)Z(p) - (Zf)(p)W(p), X(p)\rangle. \end{aligned}$$

Since $Z(p)$ and $W(p)$ are linearly independent and $X(p) \in T_PM$ was arbitrary, we conclude that both $(Wf)(p) = 0$ and $(Zf)(p) = 0$. SInce $Z(p)$ was arbitrary, f must be locally constant. Since M is connected, f is globally constant.

COLLECTIVE HOMEWORK OVER PREVIOUS WEEKS

Exercise 36.

(a) Let $\mathrm{grad}\, f(p) = \sum_{i=1}^n \alpha_i e_i$. In order to calculate the coefficients α_i, we take inner product with e_k:

$$\alpha_k = \langle \mathrm{grad}\, f(p), e_k \rangle = e_k(f).$$

This proves (a).

(b) We have

$$\mathrm{div}\,(fX)(p) = \sum_{i=1}^n \langle \nabla_{e_i} fX, e_i \rangle = \sum_{i=1}^n \langle e_i(f)X(p), e_i \rangle + f(p) \sum_{i=1}^n \langle \nabla_{e_i} X, e_i \rangle =$$

$$\langle X(p), \sum_{i=1}^n e_i(f)e_i \rangle + f(p)\mathrm{div}\, X(p) = \langle X(p), \mathrm{grad}\, f(p) \rangle + f(p)\mathrm{div}\, X(p).$$

(c) We have

$$\Delta f(p) = -\mathrm{div}\,(\sum_{i=1}^n E_i(f)E_i) = -\sum_{i=1}^n \langle \mathrm{grad}\, E_i(f)(p), e_i \rangle - \sum_{i=1}^n e_i(f)\mathrm{div}\, E_i(p)$$

$$= -\sum_{i,j=1}^n \langle e_j(E_i(f))e_j, e_i \rangle - \sum_{i=1}^n e_i(f) \sum_{j=1}^n \langle \nabla_{e_j} E_i, e_j \rangle$$

$$= -\sum_{i=1}^n \langle e_i(E_i(f)) + \sum_{i,j=1}^n e_i(f) \langle e_i, \nabla_{e_j} E_j \rangle$$

$$= -\sum_{i=1}^n \left(\langle e_i(E_i(f)) - \langle \mathrm{grad}\, f, \nabla_{e_i} E_i \rangle \right) = -\sum_{i=1}^n \left(\langle e_i(E_i(f)) - \nabla_{e_i} E_i(f) \right).$$

(d) We have

$$\Delta(fg) = -\mathrm{div}(\mathrm{grad}\,(fg)) = -\mathrm{div}(f\, \mathrm{grad}\, g) - \mathrm{div}(g\, \mathrm{grad}\, f)$$

$$= -\langle \mathrm{grad}\, f, \mathrm{grad}\, g \rangle - f\, \mathrm{div}\,(\mathrm{grad}\, g) - \langle \mathrm{grad}\, g, \mathrm{grad}\, f \rangle - g\, \mathrm{div}\,(\mathrm{grad}\, f)$$

$$= f\Delta g + g\Delta f - 2\langle \mathrm{grad}\, f, \mathrm{grad}\, g \rangle.$$

Finally, we have

$$(\Delta f, g) = \int_M (\Delta f) g\, d\mathrm{vol} = -\int_M \mathrm{div}(\mathrm{grad}\, f) g\, d\mathrm{vol}$$

$$= -\int_M \mathrm{div}(g\mathrm{grad}\, f) - \langle \mathrm{grad}\, f, \mathrm{grad}\, g \rangle\, d\mathrm{vol} = \int_M \langle \mathrm{grad}\, f, \mathrm{grad}\, g \rangle\, d\mathrm{vol} = (\mathrm{grad}\, f, \mathrm{grad}\, g).$$

The full result follows now by symmetry between f and g.

Exercise 40.

Let M be n-dimensional, and assume $K(\Sigma) = K_0$ for all 2-dimensional subspaces of TM. Then, by Exercise 39, we have

$$\langle R(v_1, v_2)v_3, v_4\rangle = K_0 \left(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\right).$$

Let $p \in M$ and $e_1, \ldots, e_n$ be an orthonormal basis of T_pM. Then

$$\begin{aligned} \mathrm{Ric}(v, w) = \sum_{i=1}^{n}\langle R(e_i, v)w, e_i\rangle &= K_0 \sum_{i=1}^{n} \left(\langle e_i, e_i\rangle\langle v, w\rangle - \langle e_i, w\rangle\langle v, e_i\rangle\right) \\ &= K_0 \left(\left(\sum_{i=1}^{n}\langle v, w\rangle\right) - \left\langle \sum_{j=1}^{n}\langle w, e_j\rangle e_j, \sum_{k=1}^{n}\langle v, e_k\rangle e_k \right\rangle\right) \\ &= K_0 \left(n\langle v, w\rangle - \langle w, v\rangle\right) = (n-1)K_0\langle v, w\rangle, \end{aligned}$$

i.e., M is an Einstein manifold with constant $(n-1)K_0$. Above, we used

$$\left\langle \sum_{j=1}^{n}\langle w, e_j\rangle e_j, \sum_{k=1}^{n}\langle v, e_k\rangle e_k \right\rangle \sum_{j,k}\langle w, e_j\rangle\langle v, e_k\rangle\delta_{jk} = \sum_{i}\langle w, e_i\rangle\langle v, e_i\rangle.$$

Exercise 43.

Since $l(s) = \int_a^b \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle^{1/2} dt$, we obtain using the Riemannian property,

$$l'(s) = \int_a^b \frac{1}{\left\|\frac{\partial F}{\partial t}(s,t)\right\|} \left\langle \frac{D}{ds}\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle.$$

Differentiating the integrand with respect to s, using the Symmetry Lemma, and setting then $s = 0$ yields

$$-\frac{1}{\|c'(t)\|^3}\left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(0,t), c'(t)\right\rangle^2 + \frac{1}{\|c'(t)\|}\frac{\partial}{\partial s}\Big|_{s=0}\left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle.$$

Using $\|c'\| = 1$ and $\frac{\partial F}{\partial s}(0,t) = X(t)$ yields

$$l''(0) = \int_a^b \left(-\left\langle \frac{D}{dt}X(t), c'(t)\right\rangle^2 + \frac{\partial}{\partial s}\Big|_{s=0}\left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle\right) dt.$$

Using, again, the Riemannian property and the Symmetry Lemma, we conclude that

$$\frac{\partial}{\partial s}\Big|_{s=0}\left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle = \left\langle \frac{D}{ds}\Big|_{s=0}\frac{D}{dt}\frac{\partial F}{\partial s}(s,t), c'(t)\right\rangle + \left\langle \frac{D}{dt}X(t), \frac{D}{dt}X(t)\right\rangle.$$

Now we make use of Lemma 7.4 to interchange the order of covariant derivatives, and obtain

$$\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0}\left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle \\
&= \left\langle R\left(\frac{\partial F}{\partial s}(0,t), \frac{\partial F}{\partial t}(0,t)\right)\frac{\partial F}{\partial s}(0,t), c'(t)\right\rangle + \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,t), c'(t)\right\rangle + \left\|\frac{D}{dt}X(t)\right\|^2 \\
&= \langle R(X(t),c'(t))X(t), c'(t)\rangle + \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,t), c'(t)\right\rangle + \left\|\frac{D}{dt}X(t)\right\|^2.
\end{aligned}$$

Now, since $X^\perp = X - \langle X, c'\rangle c'$ and c is a geodesic, we obtain

$$\frac{DX^\perp}{dt} = \frac{DX}{dt} - \left\langle \frac{DX}{dt}, c'\right\rangle c',$$

and, consequently,

$$\left\|\frac{DX^\perp}{dt}\right\|^2 = \left\|\frac{DX}{dt}\right\|^2 - 2\left\langle \frac{DX}{dt}, c'\right\rangle^2 + \left\langle \frac{DX}{dt}, c'\right\rangle^2 \|c'\|^2 = \left\|\frac{DX}{dt}\right\|^2 - \left\langle \frac{DX}{dt}, c'\right\rangle^2.$$

Putting everything together, we obtain

$$\begin{aligned}
l''(0) &= \int_a^b \left(-\left\langle \frac{D}{dt}X(t), c'(t)\right\rangle^2 + \frac{\partial}{\partial s}\Big|_{s=0}\left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(s,t)\right\rangle\right) dt \\
&= \int_a^b \left(\left\|\frac{DX^\perp}{dt}\right\|^2 + \langle R(X(t),c'(t))X(t), c'(t)\rangle + \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,t), c'(t)\right\rangle\right) dt.
\end{aligned}$$

Since $\langle R(v_1, v_2)v_3, v_4\rangle$ vanishes if $v_1 = v_2$ or $v_3 = v_4$, and $\langle R(v_1, v_2)v_3, v_4\rangle = -\langle R(v_1, v_2)v_4, v_3\rangle$, we conclude that

$$\langle R(X, c')X, c'\rangle = -\langle R(X^\perp, c')c', X^\perp\rangle$$
$$= -K(\text{span}\{X^\perp, c'\}) \left(\|X^\perp\|^2\|c'\| - \underbrace{\langle X^\perp, c'\rangle^2}_{=0} \right) = -K(\text{span}\{X^\perp, c'\})\|X^\perp\|^2.$$

Moreover, since c is a geodesic, we have

$$\int_a^b \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle dt = \int_a^b \frac{\partial}{\partial t} \left(\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle \right) dt$$
$$= \left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,b), c'(b) \right\rangle - \left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,a), c'(a) \right\rangle.$$

Since F is a proper variation, we have $\frac{\partial F}{\partial s}(s,a) = 0$ and $\frac{\partial F}{\partial s}(s,b) = 0$, and we conclude that

$$\int_a^b \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle dt = 0.$$

Combining these results, we end up with

$$l''(0) = \int_a^b \left(\left\| \frac{DX^\perp}{dt} \right\|^2 - K(\text{span}\{X^\perp, c'\})\|X^\perp\|^2 \right) dt.$$