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Riemannian Geometry IV

Solutions, set 2.

Exercise 4. We have

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial (f\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) = \frac{\partial}{\partial x_i}(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1})(\varphi(p)).$$

The last expression above is partial derivative in coordinate direction x_i of the composition of the two functions $\psi\circ\varphi^{-1} : V_1 \subset \mathbb{R}^n \to V_2 \subset \mathbb{R}^n$ and $f\circ\psi^{-1} : V_2 \subset \mathbb{R}^n \to \mathbb{R}$. The chain rule tells us that

$$\frac{\partial}{\partial x_i}(f\circ\psi^{-1}\circ\psi\circ\varphi^{-1})(\varphi(p)) = \sum_{j=1}^n \frac{\partial(f\circ\psi^{-1})}{\partial y_j}(\psi(p)) \cdot \frac{\partial(y_j\circ\varphi^{-1})}{\partial x_i}(\varphi(p)).$$

Here $\frac{\partial}{\partial y_j}$ denotes partial derivative in the j-th coordinate direction of $V_2 \subset \mathbb{R}^n$ and y_j in the expression $y_j\circ\varphi^{-1}$ denotes the j-th component function of the map ψ. So we finally end up with

$$\frac{\partial}{\partial x_i}|_p(f) = \sum_{j=1}^n \frac{\partial(y_j\circ\varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}|_p(f).$$

Exercise 5. We have $\gamma(t) = \mathbb{R}\cdot(\cos t\cos(2t), \cos t\sin(2t), \sin t)^\top$.

(a) Since

$$(\cos t\cos(2t))^2 + (\cos t\sin(2t))^2 + (\sin t)^2 = 1,$$

we obtain

$$\begin{aligned}\gamma'(0)(f) &= \frac{d}{dt}\Big|_{t=0} (\cos t\cos(2t) + \cos t\sin(2t) + \sin t)^2 \\ &= 2\cdot 3 = 6.\end{aligned}$$

(b) Let $(\gamma_1(t), \gamma_2(t)) = \varphi \circ \gamma(t)$. Then

$$\gamma_1(t) = \tan(2t) \quad \text{and}\, \gamma_2(t) = \frac{\tan t}{\cos(2t)}.$$

This implies that

$$\begin{aligned}\gamma'(t) &= \gamma_1'(t)\frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \gamma_2'(t)\frac{\partial}{\partial x_2}\Big|_{\gamma(t)} \\ &= 2(1+\tan^2(2t))\frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \frac{(1+\tan^2 t)\cos(2t) + 2\tan t\sin(2t)}{\cos^2(2t)}\frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.\end{aligned}$$