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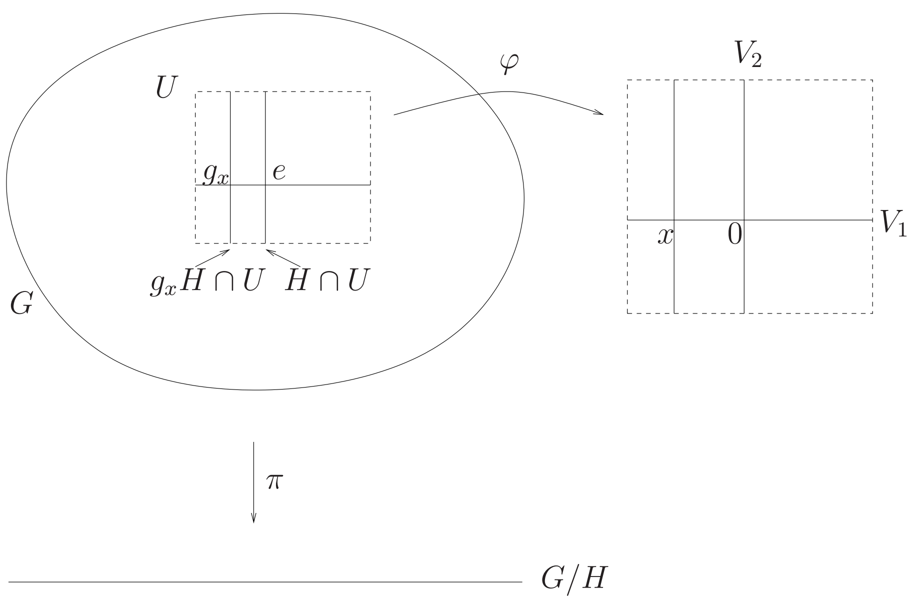
Riemannian Geometry IV

Solutions, set 3.

Note that we present the solution of Exercise 7 before the solution of Exercise 6.

Exercise 7.



(a) $\varphi = (\varphi_1, \varphi_2) : U \to V_1 \times V_2$ is a diffeomorphism. The map $\Phi : V_1 \to U$, $\Phi(x) = \varphi^{-1}((x, 0))$ is differentiable. Let $Z_e = \pi(U) \subset G/H$ and

$$F : \pi \circ \Phi : V_1 \to Z_e.$$

Z_e is obviously a neighbourhood of $eH \in G/H$. We prove that F is bijective: Assume $F(x_1) = F(x_2)$. Then $F(x_1) = g_{x_1}H = g_{x_2}H = F(x_2)$. By (c), we conclude that $x_1 = x_2$, i.e., F is injective. Let $gH \in \pi(U)$. Then there is an $x \in V_1$ such that $gH = g_x H$, and therefore $gH = F(x)$. This shows that F is surjective. Let $\Psi_e : F^{-1} : Z_e \to V_1$. Then (Ψ_e, Z_e) is a coordinate chart in a neighbourhood of $eH \in G/H$.

(b) A coordinate chart around $gH \in G/H$ is obtained from (Ψ_e, Z_e) by defining

$$(\Psi_g)^{-1} : \pi \circ L_g \circ \Phi : V_1 \to Z_g,$$

where $Z_g = \pi(gU) \subset G/H$. Bijectivity of $(\Psi_g)^{-1} : V_1 \to Z_g$ is proved as in (a). The coordinate chart is then (Ψ_g, Z_g) with $\Psi_g : Z_g \to V_1$.

(c) Let $gH \in Z_{g_1} \cap Z_{g_2}$. We have to prove that $\Psi_{g_2} \circ \Psi_{g_1}^{-1}$ is differentiable in $x_1 = \Psi_{g_1}(gH) \in V_1$. Let $x_2 = \Psi_{g_2}(gH) \in V_1$. Then $gH = g_1 g_{x_1} H = g_2 g_{x_2} H$ and we have $h \in H$ such that $g_2 g_{x_2} = g_1 g_{x_1} h$. There is an open neighbourhood $W \subset g_1 U$ of $g_1 g_{x_1}$ such that $hW \subset g_2 U$ and $\tilde{W} = \varphi(W)$ is an open neighbourhood of $(x_1, 0)$ in $V_1 \times V_2$. Then $T := \{x \in V_1 \mid (x, 0) \in \tilde{W}\}$ is an open set in V_1 containing x_1. One easily checks that the coordinate change $\Psi_{g_2} \circ \Psi_{g_1}^{-1}$ can be written as

$$\Psi_{g_2} \circ \Psi_{g_1}^{-1} = \varphi_1 \circ L_{g_2}^{-1} \circ R_h \circ L_{g_1} \circ \Phi : T \to V_1,$$

and is differentiable as a composition of differentiable maps. Since $x_1 \in T$, this shows that this coordinate change is differentiable at x_1.

Exercise 6. Let $A : (-\epsilon, \epsilon) \to SO(n)$ be a differentiable curve on the differentiable manifold $SO(n)$ with $A(0) = e$. Then we know that

$$A(t)(A(t))^\top = e,$$

for all $t \in (-\epsilon, \epsilon)$. Differentiation gives

$$A'(0)(A(0))^\top + A(0)(A'(0))^\top = A'(0)e^\top + e(A'(0))^\top = A'(0) + (A'(0))^\top = 0.$$

So we conclude that

$$T_e SO(n) \subset \{B \in M(n, \mathbb{R}) \mid B + B^\top = 0\}.$$

The right side is the space of all skew symmetric $n \times n$-matrices, which is a vector space of dimension $\frac{n(n-1)}{2}$. Since $SO(n)$ is a differentiable manifold of dimension $\frac{n(n-1)}{2}$, its tangent space $T_e SO(n)$ is a vector space of the same dimension. Since both vector spaces have the same dimension, the above inclusion is actually an equality, i.e.,

$$T_e SO(n) = \{B \in M(n, \mathbb{R}) \mid B + B^\top = 0\}.$$