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Riemannian Geometry IV

Solutions, set 6.

Exercise 14. The normal vector is given by

$$N(x_1,x_2) = \frac{\frac{\partial}{\partial x_1} \times \frac{\partial}{\partial x_2}}{\| \frac{\partial}{\partial x_1} \times \frac{\partial}{\partial x_2} \|} = (\cos x_1 \cos x_2, \cos x_1 \sin x_2, \sin x_1).$$

This implies that

$$\begin{aligned} \frac{\partial N}{\partial x_1}(x_1,x_2) &= (-\sin x_1 \cos x_2, -\sin x_1 \sin x_2, \cos x_1), \\ \frac{\partial N}{\partial x_2}(x_1,x_2) &= (-\cos x_1 \sin x_2, \cos x_1 \cos x_2, 0), \end{aligned}$$

and

$$\begin{aligned} L &= -\langle \frac{\partial N}{\partial x_1}(x_1,x_2), \frac{\partial}{\partial x_1}|_{\varphi^{-1}(x_1,x_2)} \rangle = r, \\ M &= -\langle \frac{\partial N}{\partial x_1}(x_1,x_2), \frac{\partial}{\partial x_2}|_{\varphi^{-1}(x_1,x_2)} \rangle = 0, \\ N &= -\langle \frac{\partial N}{\partial x_2}(x_1,x_2), \frac{\partial}{\partial x_2}|_{\varphi^{-1}(x_1,x_2)} \rangle = (R + r\cos x_1)\cos x_1. \end{aligned}$$

Using $E = g_{11}(\varphi^{-1}(x_1,x_2)) = r^2$, $F = g_{12}(\varphi^{-1}(x_1,x_2)) = 0$ and $G = g_{22}(\varphi^{-1}(x_1,x_2)) = (R + r\cos x_1)^2$, we conclude that

$$K(\varphi^{-1}(x_1,x_2)) = \frac{LN - M^2}{EG - F^2} = \frac{r(R + r\cos x_1)\cos x_1}{r^2(R + r\cos x_1)^2} = \frac{\cos x_1}{r(R + r\cos x_1)}.$$

$x_1 = \pi/2$ or $x_1 = 3\pi/2$ describe points of the torus intersected with the planes $Z = r$ and $Z = -r$. Note that T^2 lies between these two planes and touches each plane in a circle of radius R. Obviously, one of the principal curvatures at these points is equal to zero while the other is equal to $r > 0$, so the Gaussian curvature vanishes. $x_1 = \pi$ describes the points at the inner circle of the torus, i.e., the horizontal circle in the (X, Y)-plane with radius $R - r > 0$. These points are obvioulsy saddle points and the two principal curvatures have different signs. So the Gaussian curvature is negative at these points.

Next, we calculate

$$\int_{T^2} K\, d\text{vol} = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos x_1}{r(R + r\cos x_1)} r(R + r\cos x_1) dx_1 dx_2$$
$$= \int_0^{2\pi}\int_0^{2\pi} \cos x_1\, dx_1 dx_2 = 0.$$

The Gauss-Bonnet Theorem tells us for any closed, oriented surface $S \subset \mathbb{R}^3$ we have

$$\frac{1}{2\pi}\int_S K\, d\text{vol} = \chi(S),$$

where $\chi(S)$ is the Euler characteristic of the surface S and given by $\chi(S) = 2 - 2g$, where g is the genus of the surface. Since the genus of the torus is equal to one, we conclude that $\chi(T^2) = 0$, justifying the above calculated result.

Exercise 15. (a) We have $c'(t) = i$ for all $t \in [a, b]$. The function $l : [a, b] \to [0, L(c)]$ is given by

$$l(t) = \int_a^t \|c'(s)\|_{c(s)}\, ds = \ln\frac{t}{a}.$$

So $l : [a, b] \to [0, \ln(b/a)]$ is bijective, strictly monotone increasing and differentiable. We calculate its inverse:

$$s = l(t) \Leftrightarrow s = \ln\frac{t}{a} \Leftrightarrow e^s = \frac{t}{a} \Leftrightarrow t = ae^s.$$

So $l^{-1}(s) = ae^s$ and an arc length reparametrization of c is given by $\gamma = c \circ l^{-1} : [0, \ln(b/a)] \to \mathbb{H}^2$,

$$\gamma(s) = c(l^{-1}(s)) = c(ae^s) = ae^s i.$$

(b) We have

$$c(t) = \frac{(ai\cos t + \sin t)(ai\sin t + \cos t)}{\cos^2 t + a^2\sin^2 t} = \frac{\sin t\cos t(1 - a^2) + ia}{\cos^2 t + a^2\sin^2 t},$$

so

$$\mathrm{Im}(c(t)) = \frac{a}{\cos^2 t + a^2 \sin^2 t}.$$

On the other hand, we have

$$c'(t) = \frac{(-ai \sin t + \cos t)^2 + (ai \cos t + \sin t)^2}{(-ai \sin t + \cos t)^2} = \frac{1 - a^2}{(-ai \sin t + \cos t)^2}.$$

This implies that

$$|c'(t)| = \frac{a^2 - 1}{\cos^2 t + a^2 \sin^2 t},$$

and

$$\|c'(t)\|_{c(t)} = \frac{a^2 - 1}{\cos^2 t + a^2 \sin^2 t} \frac{\cos^2 t + a^2 \sin^2 t}{a} = \frac{a^2 - 1}{a} = a - \frac{1}{a}.$$

So we obtain

$$L(c) = \int_0^\pi \|c'(t)\|_{c(t)}\, dt = \pi \left(a - \frac{1}{a} \right).$$

Exercise 16. $c(t) = (\cos^3(t), \sin^3(t))$ implies that

$$c'(t) = 3 \sin t \cos t(-\cos(t), \sin(t)).$$

So we obtain

$$\|c'(t)\| = 3|\sin t \cos t| = \frac{3}{2} |\sin(2t)|,$$

and the length is given by

$$\begin{aligned} L(c) &= \int_0^{2\pi} \|c'(t)\| dt \\ &= \frac{3}{2} \left(\int_0^{\pi/2} \sin(2t)dt - \int_{\pi/2}^{\pi} \sin(2t)dt + \int_{\pi}^{\frac{3\pi}{2}} \sin(2t)dt - \int_{\frac{3\pi}{2}}^{2\pi} \sin(2t)dt \right) \\ &= \frac{3}{2} \cdot 4 \cdot \int_0^{\pi/2} \sin(2t)dt = 6. \end{aligned}$$

Exercise 17.

(a) If $z_1 = z_2$ there is nothing to check. Let $z_1 = ai$ and $z_2 = bi$. In this case we have

$$d_{\mathbb{H}}(z_1, z_2) = |\ln b/a|,$$

which implies that

$$\sinh(\frac{1}{2}d(z_1, z_2)) = \frac{1}{2}\Big|\exp(\frac{\ln(b/a)}{2}) - \exp(-\frac{\ln(b/a)}{2})\Big| = \frac{1}{2}\Big|\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\Big|.$$

On the other hand we obtain

$$\frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} = \frac{|a-b|}{2\sqrt{ab}} = \frac{1}{2}\Big|\frac{a}{\sqrt{ab}} - \frac{b}{\sqrt{ab}}\Big|,$$

which shows the validity of the formula in this particular case.

(b) Since f_A is an isometry (see Exercise 11), we have $d(f_A(z_1), f_A(z_2)) = d(z_1, z_2)$. This immediately implies invariance of the left hand side under f_A. As for the right-hand side note first that

$$|f_A(z_1) - f_A(z_2)| = \frac{|(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d)|}{|cz_1 + d|\,|cz_2 + d|} = \frac{|z_1 - z_2|}{|cz_1 + d|\,|cz_2 + d|}.$$

Using the identity

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2},$$

we obtain

$$\frac{|f_A(z_1) - f_A(z_2)|}{2\sqrt{\mathrm{Im}(f_A(z_1))\mathrm{Im}(f_A(z_2))}} = \frac{|z_1 - z_2|}{|cz_1 + d|\,|cz_2 + d|} \cdot \frac{|cz_1 + d|\,|cz_2 + d|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}.$$

(c) The map $f(z) = z - x$ coincides with the map f_A for $A = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. So we can apply (b). Note that the points $w_1 = f(z_1), w_2 = f(z_2)$

satisfy the requirements in (a). We conclude that

$$\begin{aligned}\sinh(\frac{1}{2}d(z_1,z_2)) &= \sinh(\frac{1}{2}d(w_1,w_2)) \qquad \text{by (b)}\\ &= \frac{|w_1-w_2|}{2\sqrt{\mathrm{Im}(w_1)\mathrm{Im}(w_2)}} \qquad \text{by (a)}\\ &= \frac{|z_1-z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \qquad \text{by (b).}\end{aligned}$$

(d) Using $c(t) = x + Re^{it}$ we calculate

$$f(c(t)) = \frac{R(e^{it}-1)}{R(e^{it}+1)} = i\frac{\sin(t/2)}{\cos(t/2)} = i\tan(t/2).$$

As t runs from 0 to π, $i\tan(t/2)$ runs from the origin along the positive imaginary axis to infinity.

Note that $f(z)$ coincides with the map f_A for

$$A = \begin{pmatrix} \frac{1}{\sqrt{2R}} & -\frac{x+r}{\sqrt{2R}} \\ \frac{1}{\sqrt{2R}} & -\frac{x-R}{\sqrt{2R}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

So we can apply (b). Note that the points $w_1 = f(z_1), w_2 = f(z_2)$ satisfy the requirements in (a). We conclude that

$$\begin{aligned}\sinh(\frac{1}{2}d(z_1,z_2)) &= \sinh(\frac{1}{2}d(w_1,w_2)) \qquad \text{by (b)}\\ &= \frac{|w_1-w_2|}{2\sqrt{\mathrm{Im}(w_1)\mathrm{Im}(w_2)}} \qquad \text{by (a)}\\ &= \frac{|z_1-z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \qquad \text{by (b).}\end{aligned}$$