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Riemannian Geometry IV

Solutions, set 7.

Exercise 18. The formula for the Christoffel symbols is given by

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(g_{il,j} + g_{jl,i} - g_{ij,l} \right).$$

In the case of the hyperbolic space we have $g_{ij}(x) = \frac{1}{x_n^2}\delta_{ij}$ and $g^{ij}(x) = x_n^2 \delta_{ij}$. Therefore the above formula simplifies to

$$\Gamma_{ij}^k = \frac{1}{2} g^{kk} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right).$$

We obviously have

$$g_{ab,c} = -\frac{2}{x_n^3} \delta_{ab} \delta_{cn}.$$

Therefore, we immediately see that $\Gamma_{ij}^k = 0$ if $i, j, k \le n-1$.

We are thus left with the cases when at least one of i, j, k is equal to n. For symmetry reasons ($\Gamma_{ij}^k = \Gamma_{ji}^k$) we only have to consider the case $i \le j$. Assume first that $k = n$. Then

$$\Gamma_{ij}^n = \frac{1}{2} x_n^2 \left(g_{in,j} + g_{jn,i} - g_{ij,n} \right).$$

If $i, j \le n-1$ we conclude that

$$\Gamma_{ij}^n = -\frac{1}{2} x_n^2 g_{ij,n} = \frac{1}{x_n} \delta_{ij}.$$

If $i \le n-1$ and $j = n$ we have

$$\Gamma_{ij}^n = 0.$$

If $i = j = k$ we finally obtain

$$\Gamma_{nn}^n = \frac{1}{2} x_n^2 g_{nn,n} = -\frac{1}{x_n}.$$

Now assume that $k \leq n-1$. Then

$$\Gamma^k_{ij} = \frac{1}{2}\left(g_{ik,j} + g_{jk,i}\right).$$

Since we already considered the case $i, j, k \leq n-1$ and we assume $i \leq j$ (by symmetry), we only have to consider $j = n$, which yields

$$\Gamma^k_{in} = \frac{1}{2}x_n^2\left(g_{ik,n} + g_{nk,i}\right) = \frac{1}{2}x_n^2\left(-\frac{2}{x_n^3}\delta_{ik} + 0\right) = -\frac{1}{x_n}\delta_{ik}.$$

Conclusion: The only non-zero Christoffel symbols are

$$\Gamma^n_{11} = \cdots = \Gamma^n_{n-1,n-1} = \frac{1}{x_n}, \quad \Gamma^n_{nn} = -\frac{1}{x_n},$$

and

$$\Gamma^1_{n1} = \Gamma^1_{1n} = \Gamma^2_{n2} = \Gamma^2_{2n} = \cdots = \Gamma^{n-1}_{n-1,n} = \Gamma^{n-1}_{n,n-1} = -\frac{1}{x_n}.$$

Not requested: The above calculations show that all covariant derivatives $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}$ vanish, unless they are of the form

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_i} = \frac{1}{x_n}\frac{\partial}{\partial x_n} \qquad \text{for } 1 \leq i \leq n-1,$$

or

$$\nabla_{\frac{\partial}{\partial x_n}}\frac{\partial}{\partial x_i} = \nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_n} = -\frac{1}{x_n}\frac{\partial}{\partial x_i} \qquad \text{for } 1 \leq i \leq n-1,$$

or

$$\nabla_{\frac{\partial}{\partial x_n}}\frac{\partial}{\partial x_n} = -\frac{1}{x_n}\frac{\partial}{\partial x_n}.$$