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Riemannian Geometry IV

Solutions, set 8.

Exercise 19. (a) Note that $T(X,Y) = -T(Y,X)$, so we only have to prove linearity in the first argument. Moreover, we obviously have $T(X_1+X_2,Y) = T(X_1,Y)+T(X_2,Y)$. We are left with showing that

$$T(fX,Y) = fT(X,Y).$$

The calculation for this goes as follows:

$$\begin{aligned} T(fX,Y) &= [fX,Y] - (\nabla_{fX}Y - \nabla_Y fX) \\ &= f[X,Y] - (Yf)X - (f\nabla_X Y - (Yf)X - f\nabla_Y X) \\ &= f([X,Y] - (\nabla_X Y - \nabla_Y X)) - (Yf)X + (Yf)X = fT(X,Y). \end{aligned}$$

(b) It is, again, straightforward to check that

$$\begin{aligned} &\nabla A(X_1,\dots,X_i+\tilde{X}_i,\dots,X_r,X_{r+1}) \\ &\qquad = \nabla A(X_1,\dots,X_i,\dots,X_r,X_{r+1}) + \nabla A(X_1,\dots,\tilde{X}_i,\dots,X_r,X_{r+1}), \end{aligned}$$

for $i = 1,2,\dots,r+1$. So it remains to show that

$$\nabla A(X_1,\dots,fX_i,\dots,X_r,X_{r+1}) = f\nabla A(X_1,\dots,X_i,\dots,X_r,X_{r+1}),$$

for $i = 1,2,\dots,r+1$. Let $i = 1,2,\dots,r$. Then

$$\begin{aligned} &\nabla A(X_1,\dots,fX_i,\dots,X_r,Y) \\ &= Y(fA(X_1,\dots,X_r)) - f\sum_{j=1}^n A(X_1,\dots,\nabla_Y X_j,\dots,X_r) - (Yf)A(X_1,\dots,X_r) \\ &\qquad = fY(A_1(X_1,\dots,X_r)) - f\sum_{j=1}^n A(X_1,\dots,\nabla_Y X_j,\dots,X_r) \\ &\qquad\qquad = f\nabla A(X_1,\dots,X_r,Y). \end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\nabla A(X_1, \ldots, X_r, fY) \\
&\qquad = fY(A(X_1, \ldots, X_2)) - \sum A(X_1, \ldots, f\nabla_Y X_j, \ldots, X_r) \\
&\qquad\qquad = f\nabla A(X_1, \ldots, X_r, Y).
\end{aligned}$$

(c) Using (b), we obtain

$$\nabla G(X, Y, Z) = Z(\langle X, Y\rangle) - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle.$$

Then $\nabla G \equiv 0$ means precisely that the affine connection ∇ has the "Riemannian property".

Exercise 20. (a) We have

$$\begin{aligned}
\frac{\partial}{\partial x_1}\Big|_{\varphi^{-1}(x_1,x_2)} &= (f'(x_1)\cos x_2, f'(x_1)\sin x_2, g'(x_1)), \\
\frac{\partial}{\partial x_2}\Big|_{\varphi^{-1}(x_1,x_2)} &= (-f(x_1)\sin x_2, f(x_1)\cos x_2, 0).
\end{aligned}$$

This implies that

$$(g_{ij}) = \begin{pmatrix} (f'(x_1))^2 + (g'(x_1))^2 & 0 \\ 0 & f^2(x_1) \end{pmatrix}$$

and

$$\left(g^{ij}\right) = \begin{pmatrix} \frac{1}{(f'(x_1))^2+(g'(x_1))^2} & 0 \\ 0 & \frac{1}{f^2(x_1)} \end{pmatrix}.$$

Consequently, the Christoffel symbols are calculated as

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}g^{11}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) = \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}, \\
\Gamma^2_{11} &= \frac{1}{2}g^{22}(g_{12,1} + g_{12,1} - g_{11,2}) = 0, \\
\Gamma^1_{12} &= \frac{1}{2}g^{11}(g_{11,2} + g_{12,1} - g_{12,1}) = 0 = \Gamma^1_{21}, \\
\Gamma^2_{12} &= \frac{1}{2}g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) = \frac{f'(x_1)}{f(x_1)} = \Gamma^2_{21}, \\
\Gamma^1_{22} &= \frac{1}{2}g^{11}(g_{12,2} + g_{12,2} - g_{22,1}) = \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}, \\
\Gamma^2_{22} &= \frac{1}{2}g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = 0.
\end{aligned}$$

This implies that

$$
\begin{aligned}
\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} &= \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{(f'(x_1))^2 + (g'(x_1))^2} \frac{\partial}{\partial x_1}, \\
\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} &= \frac{f'(x_1)}{f(x_1)} \frac{\partial}{\partial x_2}, \\
\nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} &= \frac{f'(x_1)}{f(x_1)} \frac{\partial}{\partial x_2}, \\
\nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_2} &= \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2} \frac{\partial}{\partial x_1}
\end{aligned}
$$

(b) Note that we have

$$\gamma_1'(t) = \frac{\partial}{\partial x_1}|_{\gamma_1(t)}.$$

This implies that

$$
\begin{aligned}
\left(\frac{D}{dt}\gamma_1'\right)(t) &= \nabla_{\gamma_1'(t)} \frac{\partial}{\partial x_1} = \left(\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1}\right)(\gamma_1(t)) \\
&= \frac{f'(x_1+t)f''(x_1+t) + g'(x_1+t)g''(x_1+t)}{(f'(x_1+t))^2 + (g'(x_1+t))^2} \frac{\partial}{\partial x_1}|_{\gamma_1(t)} \in T_{\gamma_1(t)}M.
\end{aligned}
$$

The condition $\frac{D}{dt}\gamma_1' \equiv 0$ is equivalent to $f'(t)f''(t) + g'(t)g''(t) = 0$ for all $t \in (a,b)$. This is equivalent to $(f'(t))^2 + (g'(t))^2 = constant$. Since

$$\|c'(t)\|^2 = (f'(t))^2 + (g'(t))^2,$$

we conclude that $\frac{D}{dt}\gamma_1'$ vanishes identically if and only if c is parametrized propertional to arc-length. Since c and γ_1 are obtained from each other by an isometry of $\mathbb{R}^3$, namely a rotation by the angle x_2 around the vertical Z-axis, c is parametrized proportional to arc-length if and only if γ_1 is parametrized proportional to arc-length. As explained in Example 25, the property $\frac{D}{dt}\gamma_1' \equiv 0$ is equivalent to the fact that γ_1 is a geodesic.

(c) We have

$$\gamma_2'(t) = \frac{\partial}{\partial x_2}|_{\gamma_2(t)}.$$

This implies that

$$\left(\frac{D}{dt}\gamma_2'\right)(t) = \nabla_{\gamma_2'(t)}\frac{\partial}{\partial x_2} = \left(\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2}\right)(\gamma_2(t))$$
$$= \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2+(g'(x_1))^2}\frac{\partial}{\partial x_1}|_{\gamma_2(t)} \in T_{\gamma_2(t)}M.$$

Since $f > 0$, the condition $\frac{D}{dt}\gamma_2' \equiv 0$ is equivalent to $f'(x_1) = 0$. Since γ_2 is a parallel of the surface of revolution M and $f_1(x_1)$ its radius, the condition $\frac{D}{dt}\gamma_2' \equiv 0$ is satisfied, e.g., if f assumes a local maximum or minimum at x_1. Again, as explained in Example 25, the property $\frac{D}{dt}\gamma_2' \equiv 0$ is equivalent to the fact that γ_2 is a geodesic.

(d) Since c is assumed to be arc-length parametrized, we have

$$(f'(x_1))^2 + (g'(x_1))^2 = 1 \qquad \text{for all } x_1 \in (a,b).$$

This implies that

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & f^2(x_1) \end{pmatrix},$$

and therefore, $\sqrt{\det(g_{ij})} = f > 0$. Since φ is an almost global coordinate chart, we conclude that

$$\mathrm{vol}(M) = \int_0^{2\pi}\int_a^b f(x_1)\,dx_1\,dx_2 = 2\pi\int_a^b f(x_1)\,dx_1.$$