A TWO-DIMENSIONAL DELTA SYMBOL METHOD AND ITS APPLICATION TO PAIRS OF QUADRATIC FORMS

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Abstract. We present a two-dimensional delta symbol method that facilitates a version of the Kloosterman refinement of the circle method, addressing a question posed by Heath-Brown. As an application, we establish the asymptotic formula for the number of integral points on a non-singular intersection of two integral quadratic forms with at least 10 variables. Assuming the Generalized Lindelöf Hypothesis, we reduce the number of variables to 9 by performing a double Kloosterman refinement. A heuristic argument suggests our two-dimensional delta symbol will typically outperform known expressions of this type by an increasing margin as the number of variables grows.

Contents

2020 *Mathematics Subject Classification.* 11P55 (11D45, 14G05, 14J45, 11D09).

Key words and phrases. Circle method, delta symbol, Diophantine equations, Hardy-Littlewood method, Kloosterman refinement, quadratic forms, exponential sum, representations of zero, Hasse principle, Manin-Peyre conjecture.

1. Overview of a delta symbol method

Consider a system of R homogeneous degree d integral forms in s variables. We aim to derive an asymptotic formula for the number of integral solutions of size at most P to this system, as P approaches infinity. A straightforward heuristic suggests that there are roughly P^{s-dR} integral solutions as long as $s > dR$. Indeed, such asymptotic formulae can be obtained using the circle method if s is sufficiently large in terms of d and R. It is an interesting question whether one can reduce the required size of s in these asymptotic formulae.

In this paper, we focus on the case when $d = R = 2$ and prove the following result for non-singular intersections of two quadrics.

Theorem 1.1. *Let F_1, F_2 be two quadratic forms with integral coefficients in $s \geq 10$ variables. Suppose the projective variety defined by $F_1(\mathbf{x}) = F_2(\mathbf{x}) = 0$ is non-singular of codimension 2. Let $w \in C_c^\infty(\mathbb{R}^s)$. Then for any $\Delta < 1/6$, we have*

$$\sum_{\substack{\mathbf{x}\in\mathbb{Z}^s \\ F_1(\mathbf{x})=F_2(\mathbf{x})=0}} w\left(\frac{\mathbf{x}}{P}\right) = \mathfrak{S}\mathfrak{I}P^{s-4} + O(P^{s-4-\Delta}), \tag{1.1}$$

where the singular series $\mathfrak{S}$ defined in (8.1) depends on F_1, F_2 and the singular integral $\mathfrak{I}$ defined in (8.2) depends on F_1, F_2 and w. The implicit constant in the error term depends on Δ, F_1, F_2 and w.

Under the Generalized Lindelöf Hypothesis (GLH) for Dirichlet L-functions, (1.1) also holds for $s = 9$ with $\Delta < 1/15$.

The asymptotic formula (1.1) verifies the Manin–Peyre conjecture for nonsingular complete intersections of two quadrics with dimension at least 7 (or 6 under GLH), improving an earlier result of Munshi [27], which requires $s \geq 11$ with $\Delta < 1/32$. Our condition $s \geq 10$ matches that in the work of Heath-Brown–Pierce [16], which handles the "split" case when $F_i(\mathbf{x}, \mathbf{y}) = G_i(\mathbf{x}) + H_i(\mathbf{y})$ where G_i, H_i are quadratic forms in at least 5 variables for $i = 1, 2$ with $\Delta < 1/32$. The conditional part of our result $s \geq 9$ matches the analogous result in the function field setting obtained by Vishe [35], where GLH follows from the generalized Riemann Hypothesis over finite fields.

Before going into details of the method, we briefly discuss what is known and conjectured for smaller s. We phrase our discussion in terms of integer solutions of homogeneous equations. (See for example [19] for more on the terminology of smooth proper models used in parts of the literature.) Let K be a number field and let $\mathscr{O}_K$ denotes the ring of integers in K. For nontrivial zeroes in $\mathscr{O}_K^s$ of two quadratic forms in s variables over K, the Hasse Principle holds provided that $s \geq 8$ and the quadratic forms cut out a smooth projective variety by a result of Heath-Brown [17], which improves upon an earlier result of Colliot-Thélène–Sansuc–Swinnerton-Dyer [9]. In a recent work, Molyakov [23] proves the Hasse Principle for $\mathscr{O}_K$-points on the smooth part of the zero locus of two quadratic forms over K, provided $s \geq 8$ and the quadratic forms define a non-conical, geometrically irreducible projective variety. If the pair of quadratic forms define a smooth codimension 2 variety and $s \geq 6$, then there is no Brauer–Manin obstruction and the Picard rank is 1. Thus the Manin–Peyre conjecture predicts that (1.1) should hold in that case, at least after removing those solutions lying on finitely many exceptional subvarieties. If we reduce the number of variables to 5, which includes the cases of del Pezzo surfaces of degree 4 and Châtelet surfaces, then the smooth Hasse principle can fail due to Brauer–Manin obstruction [9, Example 15.5].

The proof of Theorem 1.1 is based on a new version of the two-dimensional form of the circle method, which we present in Theorem 1.2 below. In order to describe this result and its context we introduce the

R-dimensional delta symbol

$$\delta_{\vec{n}} = \begin{cases} 1 & \text{if } \vec{n} = \vec{0}, \\ 0 & \text{if } \vec{n} \in \mathbb{Z}^R \setminus \{\vec{0}\}. \end{cases} \tag{1.2}$$

When $R = 1$, this is written as δ_n to detect if an integer n is zero.

1.1. **Variations of the circle method: a history.** The classical circle method from Hardy, Littlewood, Ramanujan and Vinogradov writes δ_n as $\int_0^1 e(\alpha n) d\alpha$. One then analyzes the quantity in question by breaking the unit circle into major and minor arcs. The major arc contributions are estimated asymptotically leading to the expected main terms, while the minor arc contributions are only estimated with an upper bound which should be in the error terms. This type of argument allows one to obtain the *analytic Hasse Principle* (for the exact meaning of this term see Arala [1]), when s is sufficiently large. Building on the work of Davenport [10], Birch [4] established the analytic Hasse Principle for a smooth complete intersection defined by R forms of degree d in s variables over $\mathbb{Q}$ when $s > (d-1)2^{d-1}R(R+1) + R - 1$. Improvements of Birch's result have been made in many cases. In particular, the quadratic dependency on R has been reduced to linear in the work of Rydin Myerson [29] by estimating the minor arc contributions using repulsions in exponential sums.

Another type of refinement of the circle method, aimed at reducing the number of required variables, avoids the traditional splitting of the unit circle into major and minor arcs and instead treats the contributions of all arcs asymptotically. This imitates the method of Kloosterman [18] who used the Farey dissection of the unit circle to obtain the analytic Hasse Principle for representations of integers by positive definite quaternary quadratic forms. This bypasses the barrier encountered in the Hardy–Littlewood type circle method which requires at least five variables (see also Birch [4] for $R = 1, d = 2$).

The advantage of the Kloosterman refinement is to make use of cancellations between complete exponential sums when averaged over different rationals with the same denominator. One can obtain a version of the Kloosterman refinement without appealing to the Farey dissection by using the delta symbol method, originated from the work of Duke-Friedlander-Iwaniec [11] and further developed by Heath-Brown [14]. The delta symbol method essentially provides a smooth partition of the unit circle (see Marmon-Vishe [20, Proposition 1.2]) of the form

$$\delta_n = \sum_{1 \le q \le Q} \sideset{}{^*}\sum_{a \bmod q} \int_{-Q^\delta/qQ}^{Q^\delta/qQ} p_q(w) e((a/q + w)n)\, dw + O_{\delta,N}(Q^{-N}),$$

for all $n \in \mathbb{Z}$ and all $\delta, N, Q > 0$. This not only allows one to carry out the Kloosterman refinement more easily, by switching the sum and the integral to make use of averages over the a-sum, but also allows one to take advantage of the average over the q-sum. That is, one can use cancellations in averages over rationals with different denominators, performing a *double* Kloosterman refinement, a key technique in Heath-Brown [14] for studying integral solutions to a single quadratic form in at least 3 variables. The delta symbol method has seen many other applications in recent years, such as shifted convolution problems and subconvexity estimates for L-functions (see e.g. [22, 24, 25, 26]). There are also delta symbol methods over number fields by Browning–Vishe [7] and over central simple division algebras over number fields (for example, quaternions over $\mathbb{Q}$) by Arala–Getz–Hou–Hsu–Li–Wang [2].

1.2. **The higher-dimensional problem.** It is natural to ask whether an analogue of the Kloosterman refinement of the circle method can be obtained in higher dimensions, a problem posed by Heath-Brown [14] for dimension two.

There have been various attempts of a two-dimensional Kloosterman refinement over $\mathbb{Q}$, including work of Browning–Munshi [6], Munshi [27], Heath-Brown–Pierce [16] and Arala [1] for pairs of quadratic forms and Northey–Vishe [28] for pairs of cubic forms. The work of Pierce–Schindler–Wood [31] and Browning–Pierce–Schindler [8] even extends such a process to higher dimensions. The strategy in Heath-Brown–Pierce [16], Pierce–Schindler–Wood [31] and Northey-Vishe [28] starts with the setup in the classical circle method and uses two-dimensional Dirichlet approximation with the same denominator to carry out Kloosterman refinement on the minor arc contributions. However, this is only possible when the underlying exponential sum is an absolute square since the arcs created by the Dirichlet approximation overlap.

Alternatively, one may hope to use the one dimensional delta symbol method to carry out Kloosterman refinement in higher dimensions. Browning–Munshi [6] and Arala [1] considered the special case when one equation contains a binary quadratic form independent of the rest of the variables. The structure of the binary quadratic forms allows the detection of one equation with convolutions of Dirichlet characters and the remaining equation can then be detected using the one dimensional delta symbol. More generally, Munshi [27] applied a nested delta symbol to study smooth pairs of quadratic forms. However, the moduli in these results are larger than what one expects from a two-dimensional Dirichlet approximation, thus making it less efficient in the Poisson summation step.

Progress has been made over function fields as Vishe [35] established a two-dimensional Farey dissection of the unit square over $\mathbb{F}_q(t)$, thus allowing a Kloosterman refinement in dimension two with optimal sizes of the denominators at the centers of the arcs in the dissection. This was applied to establish the analytic Hasse Principle for zero locus of any nonsingular pair of quadratic forms in at least 9 variables over $\mathbb{F}_q(t)$ when q is odd. Vishe's work has been generalized by Glas [13] to handle the nonsingular intersection of a cubic and a quadratic form over $\mathbb{F}_q(t)$. However, the non-Archimedean nature of the norm in positive characteristics was crucially used in [35], thus making it difficult to generalize the method there to the number field setting.

1.3. **A two-dimensional delta symbol method.** In this paper, we develop a two-dimensional version of the delta symbol method which provides an alternative approach to proving the analogous result of [35, Theorem 1.1] that facilitates a (double) Kloosterman refinement of the circle method in dimension two over $\mathbb{Q}$. We then apply it to study non-singular intersections of two quadratic forms over $\mathbb{Q}$ in few variables. As the heuristic in Section 12 suggests, our version of the two-dimensional delta symbol would typically outperform previous results due to the optimal size of the denominators in the center of the arcs in the smooth decomposition of the unit square.

Before we state the main results, we need to set some notation. Throughout, we will write $|\cdot|$ for the standard Euclidean norm on $\mathbb{R}^n$. Let $\delta_{\vec{n}}$ be as in (1.2). We use $A \ll_{a,\dots,z} B$ or $A = O_{a,\dots,z}(B)$ to mean that $|A| < C|B|$ for some implicit constant C depending only on the parameters $a, \dots, z$. (In later sections we slightly relax this convention, see section 2 below.) We will use $A \asymp B$ to denote $B \ll A \ll B$. We write $e(x) := \exp(2\pi i x)$ and $e_q(x) := \exp(2\pi i x/q)$. When we write a sum with an asterisk, as in $\sum^*_{\vec{a} \bmod q}$, it indicates that $\gcd(\vec{a}, q) = 1$. Given $q \in \mathbb{N}, \vec{a} \in \mathbb{Z}^2$ with $\gcd(\vec{a}, q) = 1$, a central figure in our analysis will be the lattice $\Lambda(\vec{a}, q)$ defined as

$$\Lambda(\vec{a}, q) := \{k\vec{a} + q\vec{y} : k \in \mathbb{Z}, \vec{y} \in \mathbb{Z}^2\}. \tag{1.3}$$

The following theorem summarizes our version of the two-dimensional delta symbol method.

Theorem 1.2. *Fix a smooth, L^1-normalized non-negative function ω with supported on $(1/2, 1)$. Let $\vec{\mathrm{n}} \in \mathbb{Z}^2$ and let $Q \geq 1$ be a large parameter. For each $q \in \mathbb{N}, \vec{\mathrm{a}} \in \mathbb{Z}^2$ with $\gcd(\vec{\mathrm{a}}, q) = 1$, there exists a function $p_{\Lambda(\vec{\mathrm{a}},q)}$ (defined in (4.2) below) on $\mathbb{R}^2$ depending on the choice of ω and on the lattice $\Lambda(\vec{\mathrm{a}}, q)$ such that*

$$\delta_{\vec{\mathrm{n}}} = \sum_{1\leq q\leq Q} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \int_{\mathbb{R}^2} p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}}) e((\vec{\mathrm{a}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{n}})\, d\vec{\mathrm{w}} + O_N(Q^{-N}), \tag{1.4}$$

for any $N > 0$. Here the function $p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}})$ satisfies

$$p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}}) = 1 + O_{N,\delta}(Q^{-N}) \qquad \text{if } q < Q^{1/2-\delta}, |\vec{\mathrm{w}}| < q^{-1}Q^{-1-\delta}, \tag{1.5}$$

for any $\delta > 0$. More explicitly, we have the following decomposition for $p_{\Lambda(\vec{\mathrm{a}},q)}$:

$$p_{\Lambda(\vec{\mathrm{a}},q)} = p_{1,q}(\vec{\mathrm{w}}) + \sum_{\vec{\mathrm{r}}\in\Lambda(\vec{\mathrm{a}},q)} \omega\Big(\frac{\vec{\mathrm{r}}}{Q^{1/2}}\Big) p_{2,\vec{\mathrm{r}},k,q},$$

where the functions $p_{1,q}$ and $p_{2,\vec{\mathrm{r}},k,q}$, defined in Lemma 3.2, satisfy

$$p_{1,q}(\vec{\mathrm{w}}) \ll_N \frac{Q}{q(1 + |\vec{\mathrm{w}}|Q^{3/2})}(1 + |\vec{\mathrm{w}}|qQ^{1/2})^{-N}, \tag{1.6}$$

$$p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}) \ll_N (1 + |\vec{\mathrm{w}}|kqQ^{1/2} + |\vec{\mathrm{w}} \cdot \vec{\mathrm{r}}^{\perp}|Q)^{-N}. \tag{1.7}$$

Moreover, we may interchange the sums over $\vec{\mathrm{a}} \bmod q$ and $\vec{\mathrm{r}} \in \Lambda(\vec{\mathrm{a}}, q)$ to write

$$\begin{aligned}\delta_{\vec{\mathrm{n}}} = &\sum_{1\leq q\leq Q} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \int_{\mathbb{R}^2} p_{1,q}(\vec{\mathrm{w}}) e((\vec{\mathrm{a}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{n}}) d\vec{\mathrm{w}} \\ &+ \sum_{\substack{d,k\in\mathbb{N} \\ \vec{\mathrm{c}}\in\mathbb{Z}^2 \text{primitive} \\ \vec{\mathrm{r}}=dk\vec{\mathrm{c}}}} \omega\Big(\frac{\vec{\mathrm{r}}}{Q^{1/2}}\Big) \sum_{\substack{1\leq q\leq Q/k \\ d|q \\ \gcd(q/d,k)=1}} \sideset{}{^*}\sum_{\substack{\vec{\mathrm{a}} \bmod q \\ q|d\vec{\mathrm{c}}\cdot\vec{\mathrm{a}}^{\perp}}} \int_{\mathbb{R}^2} p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}) e((\vec{\mathrm{a}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{n}}) d\vec{\mathrm{w}}.\end{aligned} \tag{1.8}$$

Remark 1.3. With an additional application of the geometry of numbers, one can show that the bound in (1.6) is also satisfied by $p_{\Lambda(\vec{\mathrm{a}},q)}$. However, we will not delve into this detail, as we use a slightly more explicit version of (1.8) stated in Proposition 5.1 below, in the application to Theorem 1.1.

Let us briefly discuss the implications of Theorem 1.2. In light of the expression

$$\delta_{\vec{\mathrm{n}}} = \int_{[0,1]^2} e(\vec{\mathrm{w}} \cdot \vec{\mathrm{n}}) d\vec{\mathrm{w}},$$

we see that (1.4) can be viewed as a smooth partition of $[0, 1]^2$ with smooth functions $p_{\Lambda(\vec{\mathrm{a}},q)}$ placed around reduced fractions $\vec{\mathrm{a}}/q$, where $q \leq Q$. The function $p_{\Lambda(\vec{\mathrm{a}},q)}$ is roughly supported in the set $\{\vec{\mathrm{a}}/q + \vec{\mathrm{w}} : |\vec{\mathrm{w}}| \ll q^{-1}Q^{-1/2+\delta}\}$. Using properties (1.5) and (1.7), one may obtain an asymptotic formula for the number of integral points on the non-singular intersection of two quadratic forms in at least 13 variables. To improve this result, we observe that the value of $p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}})$ only depends on the lattice $\Lambda(\vec{\mathrm{a}}, q)$. As shown in Lemma 4.1 below, for any $\gcd(\vec{\mathrm{a}}, q) = 1$ and $(\lambda, q) = 1$, we have $\Lambda(\vec{\mathrm{a}}, q) = \Lambda(\lambda\vec{\mathrm{a}}, q)$. Thus we can write

$$\delta_{\vec{\mathrm{n}}} = \sum_{1\leq q\leq Q} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \frac{1}{\phi(q)} \int_{\mathbb{R}^2} p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}}) \sideset{}{^*}\sum_{\lambda \bmod q} e((\lambda\vec{\mathrm{a}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{n}})\, d\vec{\mathrm{w}} + O_N(Q^{-N}).$$

This is the key property that allows us to obtain extra cancellations by averaging exponential sums on a set of the form $\{\frac{\lambda\vec{\mathrm{a}}}{q} + \vec{\mathrm{w}} : \gcd(\lambda, q) = 1\}$ for fixed $\vec{\mathrm{w}}$, thereby carrying out a version of Kloosterman refinement in dimension two. The function $p_{\Lambda(\vec{a},q)}$ is essentially supported on $|\vec{w}| \asymp \frac{1}{qQ^{1/2}}$ and thus we can choose $Q \asymp \max\{|\vec{n}|^{2/3}\}$ for applications. This size of Q is smaller than previous results and would typically be more advantageous when applying dual summation formulae.

1.4. **Outline of this paper.** We begin by setting some further notation which will be used throughout this paper in Section 2. With the key duality Lemma 3.1 and analytic properties of the p-functions in Lemma 3.2 established in Section 3, we prove Theorem 1.2 in Section 4 using properties of the lattice $\Lambda(\vec{a},q)$. We provide a heuristic comparison of Theorem 1.2 with existing δ-methods in Section 12.

The rest of the paper is dedicated to proving Theorem 1.1. In Section 5, we reduce the proof of Theorem 1.1 to Lemma 5.3 after an application of Proposition 5.1, which is essentially a restatement of Theorem 1.2. We also recall some known geometric properties of a non-singular complete intersection variety defined by two quadrics which provide guidelines for our estimations for the minor arc contributions. Section 6 is dedicated to the exponential integral bounds. We begin by re-interpreting known bounds for quadratic exponential integrals to our setting and finish with a key result Lemma 6.4, which plays a critical role in carrying out a *double* Kloosterman refinement in the case $s = 9$.

Section 7 is dedicated to the exponential sum estimates, most of which are either known or are direct analogues of their known function field counterparts. An asymptotic formula for the major arcs contribution is obtained in Section 8. After the preparations in Section 9, we prove sufficient bounds for the minor arcs contributions coming from the p_1 and p_2 functions in Section 10 and Section 11. While the final optimization in the extreme cases follows closely that in [35], the growth of our functions, for instance in (1.6), requires special care in non-extreme cases (small/medium q and $\vec{w}$).

2. Notation

We first begin by setting notation that will be used in the rest of the paper. The notation $|\cdot|$, $\delta_{\vec{n}}$, $A \ll B$, $A \asymp B$, $e(x)$, $e_q(x)$, $\sum^*_{\vec{a} \bmod q}$ and $\Lambda(\vec{a},q)$ will always be as defined before Theorem 1.2. We use $\vec{a}, \vec{b}, \vec{c}, \vec{r}$ to denote vectors in $\mathbb{Z}^2$ and $\mathbf{x}, \mathbf{u}, \ldots$ to denote vectors in $\mathbb{Z}^s$, where s will denote the number of variables needed to define the quadratic forms appearing in the statement of Theorem 1.1. Set $\mathbb{1}_S = 1$ if S is true and 0 otherwise.

We fix once and for all a function $\omega_0 \in C_c^\infty((-\frac{1}{2}, \frac{1}{2}))$ such that $\omega_0(0) = 1$ and that $\omega_0(-x) = \omega_0(x)$, and a function $\omega \in C^\infty(\mathbb{R})$ with support inside $(1/2, 1)$. We further assume that ω_0, ω take non-negative values and $\int \omega_0(x)dx = \int \omega(x)dx = 1$.

Implicit constants can depend on the choice of ω_0, ω, as they are fixed from the outset. Additionally, from section 5 onwards we fix a weight $w \in C_c^\infty(\mathbb{R}^s)$, fix a pair of quadratic forms F_1, F_2 in s variables, and use ε to represent an arbitrarily small positive constant whose value may vary from line to line. All implicit constants in $\ll, O(\cdot)$ and $\asymp$ notation are then allowed to depend on s, w, F_1, F_2 and ε.

We define

$$c = (\int_{\mathbb{R}^2} \omega(|\vec{x}|)\, d\vec{x})^{-1} = (2\pi \int_{\mathbb{R}} r\omega(r)\, dr)^{-1}. \tag{2.1}$$

Throughout this paper c always denotes this particular constant in (2.1).

Having chosen a function $\omega \in C^\infty(\mathbb{R})$ as above, we define a function

$$h(y,z) = h_\omega(y,z) = \sum_{j\in\mathbb{N}} \frac{1}{yj}\left(\omega(yj) - \omega\left(\frac{|z|}{yj}\right)\right). \tag{2.2}$$

This expression $h(y,z)$ appears in the one dimensional version of the delta symbol method and will appear in the definition of $p_{\Lambda(\vec{\mathrm{a}},q)}$. The function $h(y,z)$ depends on the function ω, which will be fixed as above except in the proof of Lemma 4.6 where $x\omega(x)$ is used in place of ω in the definition of h_2.

For each 2-vector $\vec{\mathrm{x}} = (x,y)$ we write $\vec{\mathrm{x}}^\perp = (y,-x)$. We write $\partial_{\vec{\xi}} = \frac{\xi_1}{|\vec{\xi}|}\frac{\partial}{\partial w_1} + \frac{\xi_2}{|\vec{\xi}|}\frac{\partial}{\partial w_2}$ to denote the normalized directional derivative (with respect to $\vec{\mathrm{w}}$) along $\vec{\xi}$.

Let $\Lambda \subset \mathbb{R}^2$ be a lattice, where we will typically take $\Lambda = \Lambda(\vec{\mathrm{a}},q)$. We shall study the shortest non-zero vector of the lattice $M\Lambda = \{M\vec{\mathrm{v}} : \vec{\mathrm{v}} \in \Lambda\}$. More precisely, if M is an $m \times 2$ real matrix with full rank, we define

$$\mu_M := \mu_M(\Lambda) = \min\{|M\vec{\mathrm{x}}| : \vec{\mathrm{x}} \in \Lambda \setminus \{\vec{0}\}\}.$$

In other words $\mu_M(\Lambda)$ is the Euclidean norm of the shortest nonzero vector of the lattice $M\Lambda$.

3. Two-dimensional smooth delta symbol: setup

In this section, we make some preparations to prove Theorem 1.2. We begin by elaborating the first and key step which facilitates the two-dimensional delta symbol in Lemma 3.1 below. This can be seen as a higher dimensional analogue of the equality used in the one dimensional delta symbol method of Duke-Friedlander-Iwaniec [11, (2.3)].

3.1. Detecting the delta symbol by duality of divisors.

We observe that if $\vec{\mathrm{n}} \neq \vec{0}$, then there is a unique primitive vector $\vec{\mathrm{c}}$ such that $\vec{\mathrm{n}} = \lambda\vec{\mathrm{c}}^\perp$ for some $\lambda > 0$. By symmetry, the sets $\{d : d \mid \lambda\}$ and $\{\lambda/d : d \mid \lambda\}$ are the same as long as $\lambda > 0$. The vector $\vec{\mathrm{c}}$ can be determined using the condition $\vec{\mathrm{c}} \cdot \vec{\mathrm{n}} = 0$, which can be detected with the one dimensional delta symbol. It remains to determine the size of $d, \vec{\mathrm{c}}$ that we shall use. From the two-dimensional Dirichlet approximation, which states that given any real numbers α_1, α_2 and a natural number Q, there exist integers a_1, a_2 and $1 \le q \le Q$ such that

$$\left|\alpha_i - \frac{a_i}{q}\right| \le \frac{1}{qQ^{1/2}}, \quad i = 1, 2.$$

We would like to choose $Q^{1+1/2} \asymp |\vec{\mathrm{n}}|$ so that the error terms from the approximation does not oscillate when $q \asymp Q$. Thus we hope to have a form of the two-dimensional delta symbol method that looks like

$$\frac{1}{Q}\sum_{q\le Q}\frac{1}{q^2}\sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} e_q(\vec{\mathrm{a}}\cdot\vec{\mathrm{n}})\omega\Big(\frac{|\vec{\mathrm{n}}|}{qQ^{1/2}}\Big).$$

Though we are not able to obtain this simple form of the two-dimensional delta symbol, this guides us on choosing the size of the $d, \vec{\mathrm{c}}$ parameters that will be used. Recall the notation in Section 2.

Lemma 3.1. *Let $\omega(x), h(y,z), c$ be as in Section 2. Let $\vec{\mathrm{n}} \in \mathbb{Z}^2$ and $Q \geq 1$ be a parameter, then for any $N > 0$, we have*

$$\delta_{\vec{\mathrm{n}}} = \frac{c}{Q^3} \sum_{\vec{\mathrm{r}} \in \mathbb{Z}^2} \omega\Big(\frac{|\vec{\mathrm{r}}|}{Q^{1/2}}\Big) \sum_{q \leq Q} \sideset{}{^*}\sum_{\substack{\vec{\mathrm{a}} \bmod q \\ q | \vec{\mathrm{r}} \cdot \vec{\mathrm{a}}^{\perp}}} e_q(\vec{\mathrm{a}} \cdot \vec{\mathrm{n}}) h\Big(\frac{\gcd(\vec{\mathrm{r}})q}{\gcd(q,\vec{\mathrm{r}})Q}, \frac{\vec{\mathrm{r}} \cdot \vec{\mathrm{n}}}{Q^2}\Big) - \frac{c}{Q} \sum_{d' \in \mathbb{N}} \mathbb{1}_{d' | \vec{\mathrm{n}}} \omega\Big(\frac{|\vec{\mathrm{n}}|}{d' Q^{1/2}}\Big) + O_N(Q^{-N}). \tag{3.1}$$

Proof. If $\vec{\mathrm{n}} \neq \vec{0}$, then there exists a *unique* primitive vector $\vec{\mathrm{c}}$ such that $\vec{\mathrm{n}} = \lambda \vec{\mathrm{c}}^{\perp}$ for some $\lambda > 0$. Therefore, by writing $\vec{\mathrm{n}} = dd'\vec{\mathrm{c}}^{\perp}$ we have

$$\sum_{\vec{\mathrm{c}}} \delta_{\vec{\mathrm{c}} \cdot \vec{\mathrm{n}}} \sum_{d | \vec{\mathrm{n}}} \omega\Big(\frac{d|\vec{\mathrm{c}}|}{Q_1}\Big) - \sum_{d' | \vec{\mathrm{n}}} \omega\Big(\frac{|\vec{\mathrm{n}}|}{d' Q_1}\Big) = \sum_{\vec{\mathrm{c}}} \sum_{d | \vec{\mathrm{n}}} \delta_{\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}} \omega\Big(\frac{d|\vec{\mathrm{c}}|}{Q_1}\Big) - \sum_{d' | \vec{\mathrm{n}}} \omega\Big(\frac{|\vec{\mathrm{n}}|}{d' Q_1}\Big) = 0,$$

where Q_1 is some parameter to be chosen later and the sum over $\vec{\mathrm{c}}$ is over primitive integer vectors. We apply the one dimensional delta symbol [14, Theorem 1], with $q \leq Q_2$ to detect $\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d} = 0$. This gives us

$$\delta_{\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}} = \frac{1}{Q_2^2} \sum_{q \leq Q_2} \sideset{}{^*}\sum_{a \bmod q} e_q(a\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}) h\Big(\frac{q}{Q_2}, \frac{\vec{\mathrm{c}} \cdot \vec{\mathrm{n}}}{dQ_2^2}\Big) + O_N(Q_2^{-N}),$$

for any $N > 0$, where h is defined in (2.2). Since the condition $d \mid \vec{\mathrm{n}}$ introduces a character sum with modulus d, we choose $Q_2 = Q/d \geq Q/Q_1$ so that the total moduli $qd \leq Q$, handing us

$$\sum_{\vec{\mathrm{c}}} \sum_{d | \vec{\mathrm{n}}} \delta_{\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}} \omega\Big(\frac{d|\vec{\mathrm{c}}|}{Q_1}\Big) = \sum_{\vec{\mathrm{c}}} \sum_{d | \vec{\mathrm{n}}} \omega\Big(\frac{d|\vec{\mathrm{c}}|}{Q_1}\Big) \frac{1}{Q_2^2} \sum_{q \leq Q_2} \sideset{}{^*}\sum_{a \bmod q} e_q(a\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}) h\Big(\frac{q}{Q_2}, \frac{\vec{\mathrm{c}} \cdot \vec{\mathrm{n}}}{dQ_2^2}\Big) + O_N(Q_1^2 (Q/Q_1)^{-N}),$$

for any $N > 0$. After using the additive characters to replace the condition $d \mid \vec{\mathrm{n}}$, we obtain for any primitive $\vec{\mathrm{c}}$,

$$\begin{aligned} \sum_{d | \vec{\mathrm{n}}} \sideset{}{^*}\sum_{a \bmod q} e_q(a\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}) = \frac{1}{d^2} \sideset{}{^*}\sum_{\substack{a \bmod q \\ (a,q)=1}} \sum_{\vec{\mathrm{b}} \bmod d} e_{qd}(a\vec{\mathrm{c}} \cdot \vec{\mathrm{n}} + q\vec{\mathrm{b}} \cdot \vec{\mathrm{n}}) &= \frac{1}{d^2} \sum_{\substack{\vec{\mathrm{a}} \bmod qd \\ q | \vec{\mathrm{c}} \cdot \vec{\mathrm{a}}^{\perp} \\ (\vec{\mathrm{a}},q)=1}} e_{qd}(\vec{\mathrm{a}} \cdot \vec{\mathrm{n}}) \\ &= \frac{1}{d^2} \sum_{\substack{d = d_1 d_2 \\ (d_1,q)=1}} \sum_{\substack{\vec{\mathrm{a}} \bmod qd_2 \\ q | \vec{\mathrm{c}} \cdot d_1 \vec{\mathrm{a}}^{\perp} \\ (\vec{\mathrm{a}}, qd_2)=1}} e_{qd_2}(\vec{\mathrm{a}} \cdot \vec{\mathrm{n}}), \end{aligned}$$

upon writing $d_1 = \gcd(\vec{\mathrm{a}}, d)$ and then replacing $\vec{\mathrm{a}}$ by $d_1 \vec{\mathrm{a}}$. We then re-name qd_2 as q and $\vec{\mathrm{r}} = d\vec{\mathrm{c}}$ so that $d_2 = \gcd(q,d)$, $d = \gcd(\vec{\mathrm{r}})$ and that

$$\sum_{\vec{\mathrm{c}}} \sum_{d | \vec{\mathrm{n}}} \delta_{\vec{\mathrm{c}} \cdot \frac{\vec{\mathrm{n}}}{d}} \omega\Big(\frac{d|\vec{\mathrm{c}}|}{Q_1}\Big) = \frac{1}{Q^2} \sum_{\vec{\mathrm{r}} = d\vec{\mathrm{c}}} \omega\Big(\frac{|\vec{\mathrm{r}}|}{Q_1}\Big) \sum_{q} \sideset{}{^*}\sum_{\substack{\vec{\mathrm{a}} \bmod q \\ q | \vec{\mathrm{r}} \cdot \vec{\mathrm{a}}^{\perp}}} e_q(\vec{\mathrm{a}} \cdot \vec{\mathrm{n}}) h\Big(\frac{\gcd(\vec{\mathrm{r}})q}{\gcd(q,\vec{\mathrm{r}})Q}, \frac{\vec{\mathrm{r}} \cdot \vec{\mathrm{n}}}{Q^2}\Big) + O_N(Q_1^2 (Q/Q_1)^{-N}).$$

Since we want $Q^{3/2} \asymp |\vec{\mathrm{n}}|$, we choose $Q_1 = Q^{1/2}$, making the error above $O_N(Q^{-N})$. When $\vec{\mathrm{n}} = 0$, the sum becomes

$$\sum_{d,\vec{\mathrm{c}}} \omega\Big(\frac{d|\vec{\mathrm{c}}|}{Q^{1/2}}\Big) = c^{-1} Q + O_N(Q^{-N}),$$

where c is defined in (2.1). ∎

3.2. **Defining the p-functions.** Next, we use Fourier inversion to connect the expression on the right hand side of (3.1) to the p-functions appearing in Theorem 1.2. The first sum in (3.1) can be interpreted, for every $|\vec{r}| \asymp Q^{1/2}$ and $q \le Q, \gcd(\vec{a}, q) = 1$, as placing rectangular arcs around rationals $\vec{a}/q$ on the line $q \mid \vec{r}\cdot\vec{a}^{\perp}$, which results in the term $p_{2,\vec{r},k,q}$ where $k = \gcd(\vec{r})/\gcd(\vec{r}, q)$. The second sum in (3.1) corresponds to placing symmetric arcs around each rational $\vec{a}/q$ with $q \le Q$ and $\gcd(\vec{a}, q) = 1$, which gives rise to the term $p_{1,q}$.

Lemma 3.2. *Let ω_0, ω be as in Section 2. Let $\vec{n} \in \mathbb{Z}^2$ and let $Q \ge 1$ be a large parameter. Then we have*

$$\delta_{\vec{n}} = \sum_{1\le q\le Q} \sideset{}{^*}\sum_{\vec{a} \bmod q} e_q(\vec{a}\cdot\vec{n}) \int_{\mathbb{R}^2} \Bigg(p_{1,q}(\vec{w}) + \sum_{\substack{q \mid \vec{r}\cdot\vec{a}^{\perp} \\ k=\gcd(\vec{r})/\gcd(\vec{r},q)}} \omega\Bigg(\frac{|\vec{r}|}{Q^{1/2}}\Bigg) p_{2,\vec{r},k,q}(\vec{w}) \Bigg) e(\vec{w}\cdot\vec{n})\, d\vec{w} + O_N(Q^{-N}) \tag{3.2}$$

for any $N > 0$ where

$$p_{1,q}(\vec{w}) = -\frac{c}{Q} \int_{\mathbb{R}^2} \omega_0\left(\frac{|\vec{x}|}{Q^{3/2}}\right) \sum_{j\in\mathbb{N}} \frac{1}{q^2 j^2} \omega\left(\frac{|\vec{x}|}{jqQ^{1/2}}\right) e(-\vec{w}\cdot\vec{x})\, d\vec{x}, \tag{3.3}$$

$$p_{2,\vec{r},k,q}(\vec{w}) = \frac{c}{Q^3} \int_{\mathbb{R}^2} \omega_0\left(\frac{|\vec{x}|}{Q^{3/2}}\right) h\left(\frac{kq}{Q}, \frac{\vec{r}\cdot\vec{x}}{Q^2}\right) e(-\vec{w}\cdot\vec{x})\, d\vec{x}. \tag{3.4}$$

Here $h(y, z)$ is defined by (2.2).

Proof. We will introduce a smooth weight on $\vec{n}$ in (3.1) by observing

$$\delta_{\vec{n}} = \omega_0\left(\frac{|\vec{n}|}{Q^{3/2}}\right)\delta_{\vec{n}}.$$

After using Fourier inversion in the $\vec{n}$ variable, we see that the first term on the right side in (3.1) becomes

$$\begin{aligned}
&= \frac{c}{Q^3} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{\vec{a} \bmod q} e_q(\vec{a}\cdot\vec{n}) \sum_{q\mid\vec{r}\cdot\vec{a}^{\perp}} \omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) h\left(\frac{\gcd(\vec{r})q}{\gcd(\vec{r},q)Q}, \frac{\vec{r}\cdot\vec{n}}{Q^2}\right) \omega_0\left(\frac{|\vec{n}|}{Q^{3/2}}\right) \\
&= \frac{c}{Q^3} \sum_{1\le q\le Q} \sideset{}{^*}\sum_{\vec{a} \bmod q} e_q(\vec{a}\cdot\vec{n}) \sum_{\substack{q\mid\vec{r}\cdot\vec{a}^{\perp} \\ k=\gcd(\vec{r})/\gcd(\vec{r},q)}} \omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) h\left(\frac{kq}{Q}, \frac{\vec{r}\cdot\vec{n}}{Q^2}\right) \omega_0\left(\frac{|\vec{n}|}{Q^{3/2}}\right) \\
&= \sum_{1\le q\le Q} \sideset{}{^*}\sum_{\vec{a} \bmod q} e_q(\vec{a}\cdot\vec{n}) \int_{\mathbb{R}^2} \sum_{\substack{q\mid\vec{r}\cdot\vec{a}^{\perp} \\ k=\gcd(\vec{r})/\gcd(\vec{r},q)}} \omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) p_{2,\vec{r},k,q}(\vec{w}) e(\vec{w}\cdot\vec{n})\, d\vec{w},
\end{aligned} \tag{3.5}$$

where $p_{2,\vec{r},k,q}$ is defined in (3.4). Similarly, the second term on the right side in (3.1) becomes

$$\begin{aligned}
\frac{c}{Q}\omega_0\left(\frac{|\vec{n}|}{Q^{3/2}}\right) \sum_{d'\in\mathbb{N}} \mathbb{1}_{d'\mid\vec{n}}\, \omega\left(\frac{|\vec{n}|}{d'Q^{1/2}}\right) &= \frac{c}{Q}\omega_0\left(\frac{|\vec{n}|}{Q^{3/2}}\right) \sum_{1\le q\le Q} \frac{1}{q^2} \sideset{}{^*}\sum_{\vec{a}\,(q)} e_q(\vec{a}\cdot\vec{n}) \sum_{j\in\mathbb{N}} \frac{1}{j^2}\omega\left(\frac{|\vec{n}|}{jqQ^{1/2}}\right) \\
&= -\sum_{1\le q\le Q} \sideset{}{^*}\sum_{\vec{a} \bmod q} e_q(\vec{a}\cdot\vec{n}) \int_{\mathbb{R}^2} p_{1,q}(\vec{w}) e(\vec{w}\cdot\vec{n})\, d\vec{w},
\end{aligned} \tag{3.6}$$

where $p_{1,q}$ is defined as in (3.3). The condition $q \le Q$ is introduced due to support conditions on ω_0 and ω. And the result follows from Lemma 3.1, (3.5) and (3.6). $\square$

3.3. **The function** $p_{1,q}$. We give an estimate of of $p_{1,q}(\vec{\mathrm{w}})$ defined in (3.3) in the following lemma.

Lemma 3.3. *Let* $1 \le q \le Q$ *be an integer,* $\vec{\mathrm{w}} \in \mathbb{R}^2$ *and* $p_{1,q}(\vec{\mathrm{w}})$ *be as defined in* (3.3)*. Then for any* $N > 0$,

$$p_{1,q}(\vec{\mathrm{w}}) \ll_N \frac{Q}{q(1+|\vec{\mathrm{w}}|Q^{3/2})}(1+|\vec{\mathrm{w}}|qQ^{1/2})^{-N}.$$

Proof. Making a change of variables, we write

$$p_{1,q}(\vec{\mathrm{w}}) = -c\sum_{j\in\mathbb{N}}\int_{\mathbb{R}^2}\omega_0\left(\frac{jq|\vec{\mathrm{x}}|}{Q}\right)\omega\left(|\vec{\mathrm{x}}|\right)e(-qjQ^{1/2}\vec{\mathrm{w}}\cdot\vec{\mathrm{x}})\,d\vec{\mathrm{x}}.$$

Since ω is supported in $(1/2,1)$ and ω_0 is supported in $(-1/2,1/2)$, the sum over j becomes $1 \le j \ll Q/q$. We therefore reach a trivial bound

$$|p_{1,q}(\vec{\mathrm{w}})| \ll Q/q. \tag{3.7}$$

Alternatively, upon repeated integration by parts, for a fixed $j \ll Q/q$, we obtain

$$\left|\int_{\mathbb{R}^2}\omega_0\left(\frac{jq|\vec{\mathrm{x}}|}{Q}\right)\omega\left(|\vec{\mathrm{x}}|\right)e(-qjQ^{1/2}\vec{\mathrm{w}}\cdot\vec{\mathrm{x}})\,d\vec{\mathrm{x}}\right| \ll_N (1+qjQ^{1/2}|\vec{\mathrm{w}}|)^{-N},$$

which leads to an alternate bound

$$|p_{1,q}(\vec{\mathrm{w}})| \ll_N \sum_{j\in\mathbb{N}}(1+qQ^{1/2}j|\vec{\mathrm{w}}|)^{-N} \ll_N (qQ^{1/2}|\vec{\mathrm{w}}|)^{-1}(1+qQ^{1/2}|\vec{\mathrm{w}}|)^{-N}. \tag{3.8}$$

The lemma then follows by combining (3.7) and (3.8). □

3.4. **The function** $p_{2,\vec{\mathrm{r}},k,q}$. We first give various estimates for the function $p_{2,\vec{\mathrm{r}},k,q}$ defined in (3.4). In particular, these estimates show that $p_{2,\vec{\mathrm{r}},k,q}$ is supported in a rectangle around the origin with side length $O(1/Q^{3/2})$ in the direction of $\vec{\mathrm{r}}^{\perp}$ and side length $O(1/kqQ^{1/2})$ in the direction of $\vec{\mathrm{r}}$. We then provide an asymptotic evaluation of $p_{r,\vec{\mathrm{r}},k,q}$ for certain range of $\vec{\mathrm{w}}$ in Lemma 3.6.

Lemma 3.4. *Write* $\partial_{\vec{\xi}} = \frac{\xi_1}{|\vec{\xi}|}\frac{\partial}{\partial w_1} + \frac{\xi_2}{|\vec{\xi}|}\frac{\partial}{\partial w_2}$ *for the normalized directional derivative (with respect to* $\vec{\mathrm{w}}$*). For* $Q^{1/2}/2 \le |\vec{\mathrm{r}}| \le Q^{1/2}$*, we have the following for* $a, b, j \ge 0$*:*

$$q^j\frac{\partial^j}{\partial q^j}\partial_{\vec{\mathrm{r}}}^a\partial_{\vec{\mathrm{r}}^{\perp}}^b p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}) \ll_N \mathbb{1}_{\frac{kq}{Q}<1}\left(kqQ^{1/2}\right)^a Q^{3b/2}(1+|\vec{\mathrm{w}}|kqQ^{1/2}+|\vec{\mathrm{w}}\cdot\vec{\mathrm{r}}^{\perp}|Q)^{-N}.$$

Proof. If $\frac{kq}{Q} \ge 1$, the result follows from the fact that $h(y,z) = 0$ when $y \ge 1$ and $|z| \le y/2$ (see [14, Lemma 4]) together with the support of ω and ω_0.

Next we consider $\frac{kq}{Q} < 1$. We make a change of variables $\vec{\mathrm{x}} = z_1\vec{\mathrm{r}} + z_2\vec{\mathrm{r}}^{\perp}$ to write

$$\begin{aligned} p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}) &= \frac{c}{Q^3}\int_{\mathbb{R}^2}\omega_0\left(\frac{|\vec{\mathrm{x}}|}{Q^{3/2}}\right)h\left(\frac{kq}{Q},\frac{\vec{\mathrm{r}}\cdot\vec{\mathrm{x}}}{Q^2}\right)e(-\vec{\mathrm{w}}\cdot\vec{\mathrm{x}})\,d\vec{\mathrm{x}} \\ &= \frac{c|\vec{\mathrm{r}}|^2}{Q^3}\int_{\mathbb{R}^2}\omega_0\left(\frac{|\vec{\mathrm{r}}||\vec{\mathrm{z}}|}{Q^{3/2}}\right)h\left(\frac{kq}{Q},\frac{z_1|\vec{\mathrm{r}}|^2}{Q^2}\right)e(-\vec{\mathrm{w}}\cdot(z_1\vec{\mathrm{r}}+z_2\vec{\mathrm{r}}^{\perp}))\,d\vec{\mathrm{z}}. \end{aligned} \tag{3.9}$$

Since $|\vec{r}| \asymp Q^{1/2}$ and $z_1\vec{r} + z_2\vec{r}^{\perp} = |\vec{r}|(z_1\vec{r}/|\vec{r}| + z_2\vec{r}^{\perp}/|\vec{r}|)$, we obtain

$$\begin{aligned}|q^j \frac{\partial^j}{\partial q^j} \partial_{\vec{r}}^a \partial_{\vec{r}^{\perp}}^b p_{2,\vec{r},k,q}(\vec{w})| &\ll Q^{-2} \int_{|\vec{z}|<Q^{3/2}/|\vec{r}|} (Q^{1/2} z_1)^a (Q^{1/2} z_2)^b \left| q^j \frac{\partial^j}{\partial q^j} h\left(\frac{kq}{Q}, \frac{z_1 |\vec{r}|^2}{Q^2}\right)\right| dz_1 dz_2 \\ &\ll \int_{|\vec{z}|<1} (Q^{3/2} z_1)^a (Q^{3/2} z_2)^b \left| q^j \frac{\partial^j}{\partial q^j} h\left(\frac{kq}{Q}, z_1\right)\right| dz_1 dz_2 \\ &\ll Q^{3b/2} \int_{|\vec{z}|<1} (Q^{3/2} z_1)^a \left| q^j \frac{\partial^j}{\partial q^j} h\left(\frac{kq}{Q}, z_1\right)\right| dz_1 \\ &\ll (dkqQ)^a\, Q^{2b},\end{aligned}$$

where in the last inequality we invoked [14, Lemma 5], which gives for each $R, n \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned}\frac{\partial^m}{\partial y^m}\frac{\partial^n}{\partial z^n} h(y,z) &\ll_{n,m,R} y^{-1-m-n}\Big(y^R \mathbb{1}_{n=0} + \min\{1, y^R|z|^{-R}\}\Big) \\ &\ll y^{-1-m-n} \min\{1, y^R|z|^{-R}\}, \qquad \text{if } |z| \leq 1, y<1. \end{aligned} \tag{3.10}$$

It is therefore enough to check the decay properties of $p_{2,\vec{r},k,q}$, that is, for any $A \gg 1$:

$$|q^j \frac{\partial^j}{\partial q^j} \partial_{\vec{r}}^a \partial_{\vec{r}^{\perp}}^b p_{2,\vec{r},k,q}(\vec{w})| \ll_N \left(kqQ^{1/2}\right)^a Q^{3b/2} A^{-N}, \quad \text{if } kqQ^{1/2}|\vec{w}| \gg A \text{ or } Q|\vec{w}\cdot\vec{r}^{\perp}| \gg A.$$

The condition $kqQ^{1/2}|\vec{w}| \gg A$ holds if and only if $\max\{|\vec{w}\cdot\vec{r}|, |\vec{w}\cdot\vec{r}^{\perp}|\} \gg \frac{A}{kq}$. Since $\frac{1}{kq} \geq Q^{-1}$, it is enough to consider the two cases

$$|\vec{w}\cdot\vec{r}| \gg \frac{A}{kq} \quad \text{or} \quad |\vec{w}\cdot\vec{r}^{\perp}| \gg A/Q.$$

First suppose that $|\vec{w}\cdot\vec{r}| \gg \frac{A}{kq}$. After rewriting $p_{2,\vec{r},k,q}$ in (3.9) and applying integration by parts in the z_1 variable we obtain

$$\begin{aligned}&\frac{c|\vec{r}|^2}{Q^3} q^j \frac{\partial^j}{\partial q^j} \partial_{\vec{r}}^a \partial_{\vec{r}^{\perp}}^b \int_{\mathbb{R}^2} \omega_0\left(\frac{|\vec{r}||\vec{z}|}{Q^{3/2}}\right) h\left(\frac{kq}{Q}, \frac{z_1|\vec{r}|^2}{Q^2}\right) e(-\vec{w}\cdot(z_1\vec{r} + z_2\vec{r}^{\perp}))\, d\vec{z} \\ &= \pm \frac{c|\vec{r}|^2}{Q^3} \int_{\mathbb{R}^2} \frac{e(-\vec{w}\cdot(z_1\vec{r}+z_2\vec{r}^{\perp}))}{(\vec{w}\cdot\vec{r})^N} \frac{\partial^N}{\partial z_1^N}\left[z_1^a z_2^b |\vec{r}|^{a+b} \omega_0\left(\frac{|\vec{r}||\vec{z}|}{Q^{3/2}}\right) q^j \frac{\partial^j}{\partial q^j} h\left(\frac{kq}{Q}, \frac{z_1|\vec{r}|^2}{Q^2}\right)\right] d\vec{z} \\ &\ll Q^{(a+b)/2-2} \int_{\mathbb{R}^2} \left(\frac{kq}{A}\right)^N \left| \frac{\partial^N}{\partial z_1^N}\left[z_1^a z_2^b \omega_0\left(\frac{|\vec{r}||\vec{z}|}{Q^{3/2}}\right) q^j \frac{\partial^j}{\partial q^j} h\left(\frac{kq}{Q}, \frac{z_1|\vec{r}|^2}{Q^2}\right)\right]\right| d\vec{z}.\end{aligned}$$

We invoke (3.10) again to find that above is

$$\begin{aligned}&\ll_{N,R} Q^{(a+3b)/2-2} \sum_{\substack{n_1,n_2,n_3\geq 0\\ n_1\leq a\\ N=n_1+n_2+n_3}} \int_{|\vec{z}|\ll Q} \left(\frac{kq}{A}\right)^N z_1^{a-n_1} Q^{-n_2} (kq)^{-n_3} \left(\frac{Q}{kq}\right) \min\left\{1, \frac{kq}{z_1}\right\}^R d\vec{z} \\ &\ll_{N,R} Q^{(a+3b)/2} A^{-N} \sum_{\substack{0\leq n_1\leq a\\ 0\leq n_2\leq N}} \int_{|\vec{z}|\ll 1} (kq)^{n_1+n_2} z_1^{a-n_1} Q^{a-n_1-n_2} \left(\frac{Q}{kq}\right) \min\left\{1, \frac{kq}{Qz_1}\right\}^R d\vec{z} \\ &\ll_{N,R} \left(kqQ^{1/2}\right)^a Q^{3b/2} A^{-N} \int_{\mathbb{R}^2} \omega_0(|\vec{z}|)\left(\frac{Q}{kq}\right) \min\left\{1, \frac{kq}{Qz_1}\right\}^R d\vec{z},\end{aligned}$$

where in the last line we use the fact that the term with $n_1 = n_2 = 0$ dominates the sum, $\frac{kq}{Q} < 1$ and that R can be arbitrarily large. The result follows in the case $|\vec{w}\cdot\vec{r}| \gg \frac{A}{kq}$, since the integral over $\vec{z}$ is $O(1)$.

The case $|\vec{w}\cdot\vec{r}^{\perp}| \gg A/Q$ is similar. Integration by parts in z_2 gives

$$\begin{aligned}
&\pm \frac{c|\vec{r}|^2}{Q^3} q^j \frac{\partial^j}{\partial q^j}\partial_{\vec{r}}^a \partial_{\vec{r}^\perp}^b \int_{\mathbb{R}^2} \omega_0\left(\frac{|\vec{r}||\vec{z}|}{Q^{3/2}}\right) h\left(\frac{kq}{Q}, \frac{z_1|\vec{r}|^2}{Q^2}\right) e(-\vec{w}\cdot(z_1\vec{r}+z_2\vec{r}^\perp))\, d\vec{z} \\
&= \pm Q^{(a+b)/2-2} \int_{\mathbb{R}^2} \frac{e(-\vec{w}\cdot(z_1\vec{r}+z_2\vec{r}^\perp))}{(\vec{w}\cdot\vec{r}^\perp)^N} \frac{\partial^N}{\partial z_2^N}\left[z_1^a z_2^b \omega_0\left(\frac{|\vec{r}||\vec{z}|}{Q^{3/2}}\right) q^j \frac{\partial^j}{\partial q^j} h\left(\frac{kq}{Q}, \frac{z_1|\vec{r}|^2}{Q^2}\right)\right] d\vec{z} \\
&\ll_{N,R} Q^{(a+b)/2-2} A^{-N} \sum_{\substack{0\le n_1\le b\\ n_2\ge 0\\ N=n_1+n_2}} \int_{|\vec{z}|\ll Q} Q^N z_1^a z_2^{b-n_1} Q^{-n_2} \left(\frac{Q}{kq}\right) \min\left\{1, \frac{kq}{z_1}\right\}^R d\vec{z},
\end{aligned}$$

and we can complete the argument in the same way as before. □

To further understand the function $p_{2,\vec{r},k,q}$, we need a variant of [14, Lemma 9] on the properties of the h-function.

Lemma 3.5. *Suppose $A, B, \delta > 0$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f^{(k)}(z)| \le C_k A^k$ for all $z \in \mathbb{R}$ and some sequence $\boldsymbol{C} = (C_k)_{k\ge 0}$. Then for $0 < y \le Q^{-\delta}\min\{B/A, 1\}$ we have*

$$\int_{\mathbb{R}} f(z)h(y, Bz)\, dz = B^{-1} f(0)\int_{\mathbb{R}} \omega(z)\, dz + O_{N,\delta,\boldsymbol{C}}(B^{-1}Q^{-N}).$$

Proof. Without loss of generality assume $\int_{\mathbb{R}} \omega(z)\, dz = 1$, as is the case in [14]. Making a change of variable $z' = z/B$, it is enough to consider the case when $B = 1$. Now for any $R \in \mathbb{N}$, [14, Lemma 5] gives

$$\int_{|z|>yQ^{\delta/2}} f(z)h(y,z)\, dz \ll_R C_0 \int_{|z|>yQ^{\delta/2}} y^{-1}(y^R + (y/|z|)^R)\, dz,$$

which is $O_{N,\delta,\boldsymbol{C}}(Q^{-N})$ if we choose R sufficiently large when $y \le Q^{-\delta}$. It remains to show

$$\int_{-yQ^{\delta/2}}^{yQ^{\delta/2}} f(z)h(y,z)\, dz = f(0) + O_{N,\delta,\boldsymbol{C}}(Q^{-N}). \tag{3.11}$$

Using the Taylor expansion $f(t) = f(0) + \sum_{n=1}^{2M} f_n t^n + O_{\boldsymbol{C}}((At)^{2M+1})$, we obtain

$$\begin{aligned}
\int_{-yQ^{\delta/2}}^{yQ^{\delta/2}} f(z)h(y,z)\, dz &= \sum_{n=0}^{2M} f_n \int_{-yQ^{\delta/2}}^{yQ^{\delta/2}} z^n h(y,z)\, dz + O_{\boldsymbol{C}}\left(f_{2M+1}\int_{-yQ^{\delta/2}}^{yQ^{\delta/2}} |z|^{2M+1} y^{-1}\, dz\right) \\
&= f(0) + O_{R,\boldsymbol{C}}\left(\sum_{n=0}^{2M} A^n (yQ^{\delta/2})^n \left(y^R Q^{\delta/2} + Q^{-R\delta/2}\right) + A^{2M+1}(yQ^{\delta/2})^{2M+2} y^{-1}\right),
\end{aligned}$$

by [14, Lemmas 6, 8] and $f_n \ll_n A^n C_n$ by assumption. By condition on the size of y, this becomes

$$f(0) + O_{R,\boldsymbol{C}}\left(\sum_{n=0}^{2M} (Q^{-\delta/2})^n (Q^{-R\delta+\delta/2} + Q^{-R\delta/2}) + (Q^{-\delta/2})^{2M+1} Q^{\delta/2}\right),$$

and (3.11) follows. □

Now we are ready to obtain an asymptotic formula for $p_{2,\vec{r},k,q}(\vec{w})$ for certain ranges of $\vec{w}$.

Lemma 3.6. *Let $\vec{r} \in \mathbb{Z}^2, k, q \in \mathbb{N}$ and $\delta > 0$. Suppose $|\vec{r}| \asymp Q$, $kq \le Q$, and $kq(Q|\vec{w}\cdot\vec{r}|+1) \le Q^{1-\delta}$, then we have*

$$p_{2,\vec{r},k,q}(\vec{w}) = \frac{cQ^{1/2}}{|\vec{r}|}\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{r}|}\vec{w}\cdot\vec{r}^{\perp}\Big) + O_{N,\delta}(Q^{-N}).$$

Proof. We have from (3.9) that

$$p_{2,\vec{r},k,q}(\vec{w}) = \frac{c|\vec{r}|^2}{Q^3}\int_{\mathbb{R}}\int_{\mathbb{R}}\omega_0\Big(\frac{|\vec{r}||\vec{z}|}{Q^{3/2}}\Big)h\Big(\frac{kq}{Q},\frac{z_1|\vec{r}|^2}{Q^2}\Big)e(-\vec{w}\cdot(z_1\vec{r}+z_2\vec{r}^{\perp}))dz_1dz_2.$$

The integral over z_1 can be estimated using Lemma 3.5 with

$$y = \frac{kq}{Q},\quad z = z_1,\quad A = Q^{-1}(1+Q|\vec{w}\cdot\vec{r}|),\quad B = \frac{|\vec{r}|^2}{Q^2},\quad f(z_1) = \omega_0\Big(\frac{|\vec{r}|(z_1^2+z_2^2)^{1/2}}{Q^{3/2}}\Big)e(-z_1\vec{w}\cdot\vec{r}),$$

since the hypothesis in Lemma 3.5 is guaranteed by the condition

$$kq(Q|\vec{w}\cdot\vec{r}|+1) \le Q^{1-\delta} \Leftrightarrow \frac{kq}{Q} \le Q^{-\delta}(Q|\vec{w}\cdot\vec{r}|+1)^{-1} = Q^{-\delta}B/A,$$

and the result follows after integrating over z_2. □

4. Proof of the delta symbol in Theorem 1.2

4.1. **The lattice $\Lambda(\vec{a},q)$.** To prove Theorem 1.2, we show various properties of the lattice $\Lambda(\vec{a},q)$ defined in (1.3), using the geometry of numbers. We first give some more detailed description of the lattice $\Lambda(\vec{a},q)$.

4.1.1. *Description of $\Lambda(\vec{a},q)$.* Let $q \in \mathbb{N}, \vec{a} \in \mathbb{Z}^2$ with $\gcd(\vec{a},q) = 1$. Recall the lattice $\Lambda(\vec{a},q)$ defined in (1.3). It is easy to see that

$$\mathrm{covol}(\Lambda(\vec{a},q)) = q,$$

since the lattice $\Lambda(\vec{a},q)$ is of co-dimension q in $\mathbb{Z}^2$. We begin by investigating relations among $\Lambda(\vec{a},q)$ for different $\vec{a}$.

Lemma 4.1. *Given $q \in \mathbb{N}, \vec{a} \in \mathbb{Z}^2$ with $\gcd(\vec{a},q) = 1$, we have*

(1) $\{\vec{b} \bmod q : \vec{b} \in \Lambda(\vec{a},q), \gcd(\vec{b},q) = 1\} = \{k\vec{a} \bmod q : \gcd(k,q) = 1\}$;
(2) *If* $\vec{b} \in \Lambda(\vec{a},q)$ *with* $\gcd(\vec{b},q) = 1$ *then* $\Lambda(\vec{b},q) = \Lambda(\vec{a},q)$;
(3) $\Lambda(\vec{a},q) = \{\vec{r} \in \mathbb{Z}^2 : q \mid \vec{r}\cdot\vec{a}^{\perp}\}$.

Proof. Without loss of generality, we can assume that $\gcd(\vec{a}) = 1$ since otherwise as $\gcd(\vec{a})$ and q are co-prime, we can replace $\vec{a}$ by $\vec{a}/\gcd(\vec{a})$. Write $\mathbb{Z}^2 = \mathbb{Z}\vec{a} + \mathbb{Z}\vec{a}'$. Then we have $\Lambda(\vec{a},q) = \mathbb{Z}\vec{a} + \mathbb{Z}q\vec{a}'$.

If $\gcd(k,q) = 1$ then $\gcd(k\vec{a},q) = 1$ and it follows that

$$\{k\vec{a} \bmod q : \gcd(k,q) = 1\} \subseteq \{\vec{b} \bmod q : \vec{b} \in \mathbb{Z}\vec{a} + \mathbb{Z}q\vec{a}', \gcd(\vec{b},q) = 1\}.$$

Conversely if $\vec{b} \in \mathbb{Z}\vec{a} + \mathbb{Z}q\vec{a}'$ then $\vec{b} \equiv k\vec{a} \bmod q$ for some k, and if $\gcd(\vec{b},q) = 1$ then $\gcd(k,q) = 1$. This proves the first equality, and the second equality follows since for $\gcd(k,q) = 1$,

$$\{r\vec{a} \bmod q : r \in \mathbb{Z}\} = \{rk\vec{a} \bmod q : r \in \mathbb{Z}\}.$$

For the third equality, we observe that $\vec{a}^{\perp}\cdot\vec{a}' = -\det(\vec{a}|\vec{a}') = \pm 1$ and therefore

$$\Lambda(\vec{a},q) = \mathbb{Z}\vec{a} + \mathbb{Z}q\vec{a}' = \{\vec{r} \in \mathbb{Z}\vec{a} + \mathbb{Z}\vec{a}' : q \mid \vec{r}\cdot\vec{a}^{\perp}\}.$$

□

4.1.2. *Geometry of numbers.* We state a standard result from the geometry of numbers, which can be seen as a special case of Lemma 4.1 of Maynard [21].

Lemma 4.2. *Let Λ be a lattice in $\mathbb{R}^2$. Let M be an $m \times 2$ real matrix with full rank. Further let $\vec{x}_1 \in \Lambda$ with $|M\vec{x}_1| = \mu_M(\Lambda) = \min\{|M\vec{x}| : \vec{x} \in \Lambda, \vec{x} \neq \vec{0}\}$. Then there is an $\vec{x}_2 \in \Lambda$ satisfying the following*

$$\Lambda = \mathbb{Z}\vec{x}_1 + \mathbb{Z}\vec{x}_2,$$
$$|r_1 M\vec{x}_1 + r_2 M\vec{x}_2| \asymp_m |r_1||M\vec{x}_1| + |r_2||M\vec{x}_2|, \quad (\vec{r} \in \mathbb{R}^2),$$
$$|M\vec{x}_1||M\vec{x}_2| \asymp_m \operatorname{covol}(\Lambda) \operatorname{meas}\{M[0,1]^2\}.$$

The key to bounding the contributions from the $p_{2,\vec{r},k,q}$ terms is to apply Lemma 4.2 with $\Lambda = \Lambda(\vec{a}, q)$ and a 3×2 matrix M defined as below:

Definition 4.3. Let $M = M(\vec{w})$ be the 3×2 matrix such that for any $\vec{r} \in \mathbb{Z}^2$

$$M\vec{r} = \left(\frac{r_1}{Q^{1/2}}, \frac{r_2}{Q^{1/2}}, Q\vec{w} \cdot \vec{r}^{\perp}\right)^T. \tag{4.1}$$

Lemma 4.4. *For M in defined in (4.1), we have*

$$\operatorname{meas}\left\{M[0,1]^2\right\} \asymp Q^{1/2}|\vec{w}| + Q^{-1}.$$

Proof. Let e_1, e_2, e_3 denote the standard basis for $\mathbb{R}^3$. The measure of the parallelogram $M[0,1]^2$ is equal to the norm of the vector

$$(Q^{-1/2}e_1 + Qw_2e_3) \wedge (Q^{-1/2}e_2 - Qw_1e_3) = Q^{-1}\, e_1 \wedge e_2 - Q^{1/2}w_1\, e_1 \wedge e_3 + Q^{1/2}w_2\, e_3 \wedge e_2,$$

which is

$$\sqrt{Q^{-2} + Q(w_1^2 + w_2^2)} \asymp Q^{-1} + Q^{1/2}|\vec{w}|.$$

□

4.2. **The function $p_{\Lambda(\vec{a},q)}$.** We are now ready to state the formula for $p_{\Lambda(\vec{a},q)}$ and its properties.

4.2.1. *Definition of $p_{\Lambda(\vec{a},q)}$.* After replacing the sum over $\vec{r}$ in (3.2) by $\sum_{\vec{r}\in\Lambda(\vec{a},q)}$ using Lemma 4.1, we can define $p_{\Lambda(\vec{a},q)}$ as

$$p_{\Lambda(\vec{a},q)}(\vec{w}) := p_{1,q}(\vec{w}) + \sum_{\substack{\vec{r}\in\Lambda(\vec{a},q) \\ k=\gcd(\vec{r})/\gcd(\vec{r},q)}} \omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) p_{2,\vec{r},k,q}(\vec{w}), \tag{4.2}$$

which proves (1.4).

4.2.2. *Asymptotic of $p_{\Lambda(\vec{a},q)}$.* To estimate $p_{\Lambda(\vec{a},q)}$, we first consider the sum over $\vec{r}$ in (4.2).

Lemma 4.5. *Let $\delta > 0$ and $M = M(\vec{w})$ be as in (4.1) and let $\mu_M = \mu_M(\Lambda(\vec{a},q))$ the norm of the smallest non-zero vector in $M(\Lambda(\vec{a},q))$. If*

$$\mu_M \geq Q^\delta(qQ^{-1} + |\vec{w}|qQ^{1/2}), \tag{4.3}$$

then

$$\sum_{\vec{r}\in\Lambda(\vec{a},q)} \omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) p_{2,\vec{r},k,q}(\vec{w}) = \frac{cQ^{1/2}}{q}\int_{\mathbb{R}^2} \frac{1}{|\vec{r}|}\omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right)\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{r}|}\vec{w}\cdot\vec{r}^{\perp}\Big)\, d\vec{r} + O_{N,\delta}(Q^{-N})$$

for any $N > 0$.

Proof. Throughout, we write $\vec{r} = dk\vec{c}$, where $\vec{c}$ is a primitive integer vector, $d = \gcd(\vec{r},q)$ and $\gcd(\vec{r}) = kd$ with $\gcd(q/d,k) = 1$. First we prove that

$$\sum_{\vec{r}\in\Lambda(\vec{a},q)} \omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) p_{2,\vec{r},k,q}(\vec{w}) = \sum_{\vec{r}\in\Lambda(\vec{a},q)} \frac{cQ^{1/2}}{|\vec{r}|}\omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right)\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{r}|}\vec{w}\cdot\vec{r}^{\perp}\Big) + O_{N,\delta}(Q^{-N}). \tag{4.4}$$

By Lemma 3.6, the terms on each side with $k \leq \frac{Q^{-\delta/2}}{|\vec{w}|qQ^{1/2}+qQ^{-1}}$ agree up to negligible error. We show that on each side of (4.4), there is only a negligible contribution from values of $\vec{r}$ with

$$k \geq \frac{Q^{-\delta/2}}{|\vec{w}|qQ^{1/2}+qQ^{-1}}. \tag{4.5}$$

Suppose that $\vec{r}$ satisfies (4.5). It is easy to see that $\frac{\vec{r}}{k} = d\vec{c} \in \Lambda(\vec{a},q)$, and therefore

$$|M(d\vec{c})| \geq \mu_M.$$

Together with (4.5) and the asumption on μ_M in the lemma, this yields

$$\frac{|\vec{r}|}{Q^{1/2}} + Q\vec{w}\cdot\vec{r}^{\perp} \asymp |M\vec{r}| \geq \mu_M k \geq Q^{\delta/2}.$$

Now using Lemma 3.4, one can check that for any such $\vec{r}$, we have

$$\omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right) p_{2,\vec{r},k,q}(\vec{w}) \ll_{N,\delta} Q^{-N},$$

$$\omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right)\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{r}|}\vec{w}\cdot\vec{r}^{\perp}\Big) \ll_{N,\delta} Q^{-N},$$

as required, verifying (4.4).

We next use the Poisson summation formula to prove that the sum on the right hand side of (4.4) may be replaced by an integral up to an admissible error. Let $X = (\vec{x}_1|\vec{x}_2)$ be the matrix with columns given by the vectors $\vec{x}_i$ from Lemma 4.2 for the lattice $\Lambda = \Lambda(\vec{a},q)$ and M. By (4.4) it suffices to prove

$$\sum_{\vec{r}\in\Lambda(\vec{a},q)} \frac{cQ^{1/2}}{|\vec{r}|}\omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right)\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{r}|}\vec{w}\cdot\vec{r}^{\perp}\Big) = \frac{cQ^{1/2}}{q}\int_{\mathbb{R}^2} \frac{1}{|\vec{r}|}\omega\left(\frac{|\vec{r}|}{Q^{1/2}}\right)\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{r}|}\vec{w}\cdot\vec{r}^{\perp}\Big)\, d\vec{r} + O_{N,\delta}(Q^{-N}).$$

We can re-write the left-hand side above as

$$\sum_{\vec{y}\in\mathbb{Z}^2} W(MX\vec{y}), \text{ where } W((x,y,z)^T) = \frac{c}{|\binom{x}{y}|}\omega\big(|\tbinom{x}{y}|\big)\widehat{\omega}_0\big(|\tbinom{x}{y}|^{-1}z\big).$$

By Lemma 4.2 and the fact that $\mathrm{covol}(\Lambda(\vec{\mathrm{a}},q))=q$, we have

$$|M\vec{\mathrm{x}}_1||\vec{\mathrm{y}}|\ll|M\vec{\mathrm{x}}_1||y_1|+|M\vec{\mathrm{x}}_2||y_2|\asymp|MX\vec{\mathrm{y}}|\ll|M\vec{\mathrm{x}}_2||\vec{\mathrm{y}}|\ll\frac{q\,\mathrm{meas}\{M[0,1]^2\}}{|M\vec{\mathrm{x}}_1|}|\vec{\mathrm{y}}|.$$

From the assumption $\mu_M=|M\vec{\mathrm{x}}_1|\geq Q^{\delta}(qQ^{1/2}|\vec{\mathrm{w}}|+qQ^{-1})$ combined with Lemma 4.4, we obtain

$$|MX\vec{\mathrm{y}}|\ll Q^{-\delta}|\vec{\mathrm{y}}|.$$

Using this bound, a routine application of Poisson summation and summation by parts shows that

$$\sum_{\vec{\mathrm{y}}\in\mathbb{Z}^2}W(MX\vec{\mathrm{y}})=\int_{\mathbb{R}^2}W(MX\vec{\mathrm{y}})\,d\vec{\mathrm{y}}+O_{N,\delta}(Q^{-N}),$$

which concludes the result. □

Next we apply Lemma 4.5 to prove that under the assumption (4.3), the function $p_{\Lambda(\vec{\mathrm{a}},q)}$ is equal to 1, up to a very small error. This result will imply (1.5).

Lemma 4.6. *Let $M=M(\vec{\mathrm{w}})$ and $\mu_M=\mu_M(\Lambda(\vec{\mathrm{a}},q))$ be as in Lemma 4.5. Let $1\leq q\leq Q$ and $\delta>0$. If*

$$\mu_M\geq Q^{\delta}(qQ^{-1}+|\vec{\mathrm{w}}|qQ^{1/2}),$$

then

$$p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}})=1+O_{N,\delta}(Q^{-N}),$$

for any $N>0$.

Proof. By Lemma 4.5, and making a change of variables using $\vec{\mathrm{x}}'=z\vec{\mathrm{r}}/|\vec{\mathrm{r}}|, z'=|\vec{\mathrm{x}}|$, we obtain

$$\begin{aligned}
p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}})-p_{1,q}(\vec{\mathrm{w}})&=\frac{cQ^{1/2}}{q}\int_{\mathbb{R}^2}\frac{1}{|\vec{\mathrm{r}}|}\omega\left(\frac{|\vec{\mathrm{r}}|}{Q^{1/2}}\right)\widehat{\omega}_0\Big(\frac{Q^{3/2}}{|\vec{\mathrm{r}}|}\vec{\mathrm{w}}\cdot\vec{\mathrm{r}}^{\perp}\Big)\,d\vec{\mathrm{r}}+O_{N,\delta}(Q^{-N})\\
&=\frac{1}{q}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\frac{cQ^{1/2}}{|\vec{\mathrm{r}}|}\omega\left(\frac{|\vec{\mathrm{r}}|}{Q^{1/2}}\right)\omega_0(z)e\left(-zQ^{3/2}\vec{\mathrm{w}}\cdot\frac{\vec{\mathrm{r}}^{\perp}}{|\vec{\mathrm{r}}|}\right)\,dz\,d\vec{\mathrm{r}}+O_{N,\delta}(Q^{-N})\\
&=\frac{1}{q}\int_{\mathbb{R}^2}\int_{\mathbb{R}}\frac{cQ^{1/2}}{|\vec{\mathrm{x}}'|}\omega\left(\frac{z'}{Q^{1/2}}\right)\omega_0(|\vec{\mathrm{x}}'|)e\left(-Q^{3/2}\vec{\mathrm{w}}\cdot\vec{\mathrm{x}}'^{\perp}\right)\,dz'\,d\vec{\mathrm{x}}'+O_{N,\delta}(Q^{-N})\\
&=\frac{cQ}{q}\int_{\mathbb{R}^2}\frac{1}{|\vec{\mathrm{x}}'|}\omega_0(|\vec{\mathrm{x}}'|)e\left(-Q^{3/2}\vec{\mathrm{w}}\cdot\vec{\mathrm{x}}'^{\perp}\right)\,d\vec{\mathrm{x}}'+O_{N,\delta}(Q^{-N}),
\end{aligned}$$

for any $N>0$. By Lemmas 4.2 and 4.4, we have

$$\mu_M\ll(qQ^{-1}+|\vec{\mathrm{w}}|qQ^{1/2})^{1/2}. \tag{4.6}$$

Therefore, by the assumption on μ_M we have $q\ll Q^{1-\delta}$, and so we can re-write the above after a substitution $\vec{\mathrm{x}}=Q^{3/2}\vec{\mathrm{x}}'$ as

$$p_{\Lambda(\vec{\mathrm{a}},q)}(\vec{\mathrm{w}})-p_{1,q}(\vec{\mathrm{w}})=\frac{c}{Q^{3/2}}\left(\sum_{j\in\mathbb{N}}\omega\left(\frac{qj}{Q}\right)\right)\int_{\mathbb{R}^2}\frac{1}{|\vec{\mathrm{x}}|}\omega_0\left(\frac{|\vec{\mathrm{x}}|}{Q^{3/2}}\right)e\left(-\vec{\mathrm{w}}\cdot\vec{\mathrm{x}}^{\perp}\right)\,d\vec{\mathrm{x}}+O_{N,\delta}(Q^{-N}).$$

After plugging in the definition of $p_{1,q}(\vec{w})$ defined in (3.3), we can write

$$p_{\Lambda(\vec{a},q)}(\vec{w}) = \frac{c}{Q^{3/2}} \int_{\mathbb{R}^2} \left(\sum_{j\in\mathbb{N}} \omega\left(\frac{qj}{Q}\right) - \frac{Q^{1/2}|\vec{x}|}{j^2q^2}\omega\left(\frac{|\vec{x}|}{jqQ^{1/2}}\right)\right) \frac{1}{|\vec{x}|}\omega_0\left(\frac{|\vec{x}|}{Q^{3/2}}\right) e\left(-\vec{w}\cdot\vec{x}^{\perp}\right) d\vec{x} + O_{N,\delta}(Q^{-N}),$$

$$= cQ^{-3/2} \int_{\mathbb{R}^2} \frac{1}{|\vec{x}|}\omega_0\left(\frac{|\vec{x}|}{Q^{3/2}}\right) h_2\left(\frac{q}{Q}, \frac{|\vec{x}|}{Q^{3/2}}\right) e(-\vec{w}\cdot\vec{x})\, d\vec{x} + O_{N,\delta}(Q^{-N}),$$

where

$$h_2(y,z) = \sum_{j\in\mathbb{N}} \frac{1}{yj}\left(yj\omega\left(yj\right) - \frac{z}{yj}\omega\left(\frac{z}{yj}\right)\right).$$

Using the assumption in the lemma and (4.6), we have $qQ^{-1} + |\vec{w}|qQ^{1/2} \ll Q^{-2\delta}$, so that we can apply Lemma 3.5 with

$$h = h_{x\omega(x)}, \quad y = \frac{q}{Q}, \quad A = Q^{-3/2} + |\vec{w}|, \quad B = Q^{-3/2}, \quad z = |\vec{x}|,$$

and conclude that

$$p_{\Lambda(\vec{a},q)}(\vec{w}) = c\int_0^{2\pi}\int_0^{\infty} x\omega(x)\, dx\, d\theta + O_{N,\delta}(Q^{-N}) = 1 + O_{N,\delta}(Q^{-N}),$$

which proves the lemma. □

4.3. **Proof of Theorem 1.2.** Equation (1.4) follows from (4.2). Applying Lemma 4.6 and noting that $\mu_M \geq Q^{-1/2}$, we obtain (1.5). The bound (1.6) follows from Lemma 3.3 and (1.7) follows from Lemma 3.4. (1.8) follows from splitting $p_{\Lambda(\vec{a}.q)}(\vec{w})$ into two sums using (4.2), switching the order of summation, separating the two sums and in the second sum, upon further writing $\vec{r} = kd\vec{c}$ with $\vec{c} \in \mathbb{Z}^2$ primitive, $\gcd(\vec{r},q) = d$, so that $\gcd(q/d,k) = 1$.

5. Applications to rational points: the setup

5.1. **The major and minor arcs and a version of Theorem 1.2.** To state our more explicit version of (1.4), we introduce some notation. Let $\delta > 0$ be sufficiently small. In view of the properties (1.5) and (1.7), we define the major arcs as

$$\mathfrak{M}_q = \mathfrak{M}_q(\delta) = \left\{\vec{w} \in \mathbb{R}^2 : |\vec{w}| < q^{-1}Q^{-1-\delta}\right\} \qquad \text{if } 1 \leq q \leq Q^{1/2-\delta},$$

and the minor arcs as

$$\mathfrak{m}_q = \mathfrak{m}_q(\delta) = \begin{cases} \left\{\vec{w} \in \mathbb{R}^2 : q^{-1}Q^{-1-\delta} \leq |\vec{w}| < q^{-1}Q^{-1/2+\delta}\right\} & \text{if } 1 \leq q \leq Q^{1/2-\delta}, \\ \left\{\vec{w} \in \mathbb{R}^2 : |\vec{w}| < q^{-1}Q^{-1/2+\delta}\right\} & \text{if } Q^{1/2-\delta} < q \leq Q. \end{cases} \tag{5.1}$$

Most authors do not take major and minor arcs in the δ-method, and indeed they are a little different to the classical case. Note that for us the arcs $\mathfrak{M}_q, \mathfrak{m}_q$ are all centred at $\vec{0}$. The traditional (shifted) major and minor arcs would be $\vec{a}/q + \mathfrak{M}_q$ and $\vec{a}/q + \mathfrak{m}_q$ for $(\vec{a},q) = 1$. Contrary to the classical circle method, here the union of the shifted major arcs $\mathfrak{M} = \bigcup_{q=1}^{Q^{1/2-\delta}} \bigcup_{\vec{a}}^{*}(\vec{a}/q + \mathfrak{M}_q)$ is not disjoint from the union of the shifted minor arcs $\mathfrak{m} = \bigcup_{q=1}^{Q} \bigcup_{\vec{a}}^{*}(\vec{a}/q + \mathfrak{m}_q)$. This is acceptable because, on each arc, we integrate with respect to a different kernel function $p_{\Lambda(\vec{a},q)}$.

We begin by noting that using (1.5), on major arcs the function $p_{\Lambda(\vec{a},q)}$ is equal to 1 with a very small error. On the minor arcs, it would be preferable for us to not work with the kernel function $p_{\Lambda(\vec{a},q)}$ directly.

We will instead apply the following proposition which is an easy corollary of Theorem 1.2. In particular, it follows from a combination of (1.5) and (1.8). This would allow us to work with known exponential sums and also draw parallels with [35, Theorem 1.1] and further allow us to re-use some of the bounds obtained in [35] over minor arcs. More precisely, the sum E_2 appearing in Proposition 5.1 will lead to the term $N_2(P,\delta)$ appearing (5.4) which corresponds closely with [35, Lemma 5.1].

Proposition 5.1. *Let $\vec{\mathrm{n}} \in \mathbb{Z}^2$ and let $Q \geq 1$ be a large parameter. Then for any $\delta, N > 0$ with δ sufficiently small, we have*

$$\delta_{\vec{\mathrm{n}}} = M + E_1 + E_2 + O_{N,\delta}(Q^{-\delta N}),$$

where

$$M = \sum_{1\leq q\leq Q^{1/2-\delta}} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \int_{\vec{\mathrm{w}}\in\mathfrak{M}_q} e((\vec{\mathrm{a}}/q+\vec{\mathrm{w}})\cdot\vec{\mathrm{n}})d\vec{\mathrm{w}},$$

$$E_1 = \sum_{1\leq q\leq Q} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \int_{\vec{\mathrm{w}}\in\mathfrak{m}_q} p_{1,q}(\vec{\mathrm{w}}) e((\vec{\mathrm{a}}/q+\vec{\mathrm{w}})\cdot\vec{\mathrm{n}})d\vec{\mathrm{w}},$$

$$E_2 = \sum_{\substack{d,k\in\mathbb{N}\\ \vec{\mathrm{c}}\in\mathbb{Z}^2\text{primitive}\\ \vec{\mathrm{r}}=dk\vec{\mathrm{c}}}} \omega\Big(\frac{\vec{\mathrm{r}}}{Q^{1/2}}\Big) \sum_{\substack{1\leq q\leq Q/k\\ d|q\\ \gcd(q/d,k)=1}} \sideset{}{^*}\sum_{\substack{\vec{\mathrm{a}} \bmod q\\ q|d\vec{\mathrm{c}}\cdot\vec{\mathrm{a}}^{\perp}}} \int_{\vec{\mathrm{w}}\in\mathfrak{m}_q} p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}) e((\vec{\mathrm{a}}/q+\vec{\mathrm{w}})\cdot\vec{\mathrm{n}})d\vec{\mathrm{w}},$$

with ω as in Theorem 1.2, functions $p_{1,q}$ and $p_{2,\vec{\mathrm{r}},k,q}$ as defined as in Lemma 3.2, satisfying decay properties (1.6) and (1.7).

Let $\vec{\mathrm{F}} = (F_1, F_2)$ be a pair of forms of degree $d \geq 2$. Let $P \geq 1$ be a large parameter, let $w \in C_c^\infty(\mathbb{R}^s)$ be a smooth function with a compact support. We consider the following counting function

$$N(P) := N_{\vec{\mathrm{F}},w}(P) := \sum_{\substack{\mathbf{x}\in\mathbb{Z}^s\\ \vec{\mathrm{F}}(\mathbf{x})=\vec{0}}} w(\mathbf{x}/P) = \sum_{\mathbf{x}\in\mathbb{Z}^s} w(\mathbf{x}/P)\delta_{\vec{\mathrm{F}}(\mathbf{x})}.$$

We choose Q such that $Q^{1+1/2} = P^d$. We only consider the case $d = 2$ in the following application and so we set $Q = P^{4/3}$ from now on. Applying Proposition 5.1, we write

$$N(P) = N_0(P,\delta) + N_1(P,\delta) + N_2(P,\delta) + O_{N,\delta}(Q^{-N\delta}), \tag{5.2}$$

where

$$N_0(P,\delta) := \sum_{1\leq q<Q^{1/2-\delta}} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \sum_{\mathbf{x}\in\mathbb{Z}^s} e_q(\vec{\mathrm{a}}\cdot\vec{\mathrm{F}}(\mathbf{x})) \int_{\vec{\mathrm{w}}\in\mathfrak{M}_q(\delta)} w(\mathbf{x}/P) e(\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))\, d\vec{\mathrm{w}},$$

$$N_1(P,\delta) := \sum_{1\leq q\leq Q} \sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \sum_{\mathbf{x}\in\mathbb{Z}^s} e_q(\vec{\mathrm{a}}\cdot\vec{\mathrm{F}}(\mathbf{x})) \int_{\vec{\mathrm{w}}\in\mathfrak{m}_q(\delta)} p_{1,q}(\vec{\mathrm{w}}) w(\mathbf{x}/P) e(\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))\, d\vec{\mathrm{w}}, \tag{5.3}$$

$$\begin{aligned} N_2(P,\delta) := \sum_{\substack{d,k\in\mathbb{N}\\ \vec{\mathrm{c}}\text{ primitive}\\ \vec{\mathrm{r}}=dk\vec{\mathrm{c}}}} \omega\Big(\frac{\vec{\mathrm{r}}}{Q^{1/2}}\Big) \sum_{\substack{1\leq q\leq Q/k\\ d|q\\ \gcd(q/d,k)=1}} \sideset{}{^*}\sum_{\substack{\vec{\mathrm{a}} \bmod q\\ q|d\vec{\mathrm{c}}\cdot\vec{\mathrm{a}}^{\perp}}} \sum_{\mathbf{x}\in\mathbb{Z}^s} w(\mathbf{x}/P) e_q(\vec{\mathrm{a}}\cdot\vec{\mathrm{F}}(\mathbf{x}))& \\ \times \int_{\vec{\mathrm{w}}\in\mathfrak{m}_q(\delta)} p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}) e(\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))\, d\vec{\mathrm{w}}.& \end{aligned} \tag{5.4}$$

By applying Poisson summation in the $\mathbf{x}$ variable with modulus q, we obtain the following lemma.

Lemma 5.2. *With the notation above, we have*

$$N_0(P,\delta)=\sum_{1\le q<Q^{1/2-\delta}} q^{-s}\sum_{\mathbf{u}\in\mathbb{Z}^s} D_q(\mathbf{u})\int_{\vec{\mathrm{w}}\in\mathfrak{M}_q(\delta)} I_q(\vec{\mathrm{w}},\mathbf{u})\,d\vec{\mathrm{w}},$$

$$N_1(P,\delta)=\sum_{1\le q<Q} q^{-s}\sum_{\mathbf{u}\in\mathbb{Z}^s} D_q(\mathbf{u})\int_{\mathfrak{m}_q(\delta)} p_{1,q}(\vec{\mathrm{w}})I_q(\vec{\mathrm{w}},\mathbf{u})\,d\vec{\mathrm{w}},$$

$$N_2(P,\delta)=\sum_{\substack{d,k\in\mathbb{N}\\ \vec{\mathrm{c}}\in\mathbb{Z}^2\text{primitive}\\ \vec{\mathrm{r}}=dk\vec{\mathrm{c}}}} \omega\Big(\frac{\vec{\mathrm{r}}}{Q^{1/2}}\Big)\sum_{\substack{1\le q\le Q/k\\ d|q\\ \gcd(q/d,k)=1}} q^{-s}\sum_{\mathbf{u}\in\mathbb{Z}^s} S_{q,d\vec{\mathrm{c}}}(\mathbf{u})\int_{\vec{\mathrm{w}}\in\mathfrak{m}_q(\delta)} p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}})I_q(\vec{\mathrm{w}},\mathbf{u})\,d\vec{\mathrm{w}},$$

where the exponential sums $D_q, S_{q,d\vec{\mathrm{c}}}$ and the exponential integral I_q are defined as

$$D_q(\mathbf{u})=\sideset{}{^*}\sum_{\vec{\mathrm{a}}\bmod q}\sum_{\mathbf{b}\bmod q} e_q(\vec{\mathrm{a}}\cdot\vec{\mathrm{F}}(\mathbf{b})+\mathbf{b}\cdot\mathbf{u}), \tag{5.5}$$

$$S_{q,d\vec{\mathrm{c}}}(\mathbf{u})=\sideset{}{^*}\sum_{\substack{\vec{\mathrm{a}}\bmod q\\ q|d\vec{\mathrm{c}}\cdot\vec{\mathrm{a}}^{\perp}}}\sum_{\mathbf{b}\bmod q} e_q(\vec{\mathrm{a}}\cdot\vec{\mathrm{F}}(\mathbf{b})+\mathbf{b}\cdot\mathbf{u}), \tag{5.6}$$

$$I_q(\vec{\mathrm{w}},\mathbf{u})=\int_{\mathbb{R}^s} w\left(\frac{\mathbf{x}}{P}\right)e(\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e_q(-\mathbf{x}\cdot\mathbf{u})d\mathbf{x}. \tag{5.7}$$

By investigating the exponential sum $D_q, S_{q,d\vec{\mathrm{c}}}$ and the exponential integral I_q, we obtain the following lemma which immediately proves Theorem 1.1.

Lemma 5.3. *Let $\varepsilon>0$ be any constant. With the notation above, for all sufficiently small δ (depending on s and ε) we have*

$$\begin{aligned} N_0(P,\delta)-P^{s-4}\mathfrak{S}\mathfrak{J} &\ll P^{s-4-1/3+\varepsilon},\\ N_1(P,\delta) &\ll P^{s-4-(s-8)/3+\varepsilon},\\ N_2(P,\delta) &\ll P^{s-4-(s-10+1/2)/3+\varepsilon} \text{ for } s\ge 10,\\ N_2(P,\delta) &\ll P^{s-4-(s-9+1/5)/3+\varepsilon} \text{ under GLH for } s\ge 9. \end{aligned}$$

Here all the implicit constants may depend on s, w, F_1, F_2 and ε.

Proof of Theorem 1.1. Combining (5.2) with Lemma 5.3, we obtain the result by choosing δ sufficiently small. ∎

5.2. **Geometry of pairs of quadratic forms.** To estimate the exponential sums and exponential integrals, we need some properties of the geometry of non-singular intersection of two quadrics. The reader may consult the summary of Browning–Munshi [6, §2.2] as well as Heath-Brown–Pierce [16, Section 2]. Suppose $\vec{\mathrm{F}}=(F_1,F_2)$ is a pair of quadratic forms in s variables such that the projective variety [1] defined by $F_1(\mathbf{x})=F_2(\mathbf{x})=0$ is non-singular over $\overline{\mathbb{Q}}$. Let M_1 and M_2 denote integer matrices defining the quadratic forms F_1 and F_2. This projective variety has a dual variety, which is an absolutely irreducible hypersurface of degree $4(s-2)$ when $s\ge 4$. (See [6, §2.2] for more details; the proof implicitly requires the absolute

[1]By a variety over K, we mean a *reduced* separated scheme of finite type over Spec K. In fact, all varieties occurring here will be quasiprojective; thus the reader is free to interpret *variety* as an *open subset of a reduced, not necessarily irreducible, projective variety.*

irreducibility of the smooth complete intersection $F_1 = F_2 = 0$, which holds for $s \geq 4$, and the result of Aznar referred to there is a composite of [3, Theorems 2-3].)

In what follows we will therefore take the dual variety to be defined by $\mathscr{F}^*(\mathbf{u}) = 0$, where $\mathscr{F}^*$ is a homogeneous polynomial with integer coefficients of degree $4(s-2)$ which is irreducible over $\overline{\mathbb{Q}}$. Given a primitive integer vector $\vec{c}$, let $F_{\vec{c}}$ be the quadratic form

$$F_{\vec{c}} := c_1F_1 + c_2F_2,$$

defined by the integer matrix

$$M_{\vec{c}} := c_1M_1 + c_2M_2.$$

Let $F(x,y) = \det(xF_1 + yF_2)$ and $D_F = 2\mathrm{Disc}(F)$ where $\mathrm{Disc}(F)$ denotes the discriminant of $F(x,y)$ as a binary form.

Over the splitting field K we have

$$\det(xF_1 + yF_2) = h^{-1}\prod_{i=1}^{s}(\lambda_i x + \mu_i y),$$

for some $h \in \mathbb{N}, \lambda_i, \mu_i \in \mathscr{O}_K$.

Let $S, T \in \mathrm{GL}(s, \mathbb{Z})$ be invertible integral matrices appearing in a Smith normal form of the matrix $M_{\vec{c}} = TDS$ where D is a diagonal integer matrix with entries $\rho_1|\cdots|\rho_s$. For $\det(M_{\vec{c}}) \neq 0$, let $F^*_{\vec{c}}$ denote the dual form for the quadratic form $F_{\vec{c}}$ defined by

$$F^*_{\vec{c}}(\mathbf{u}) := \det(M_{\vec{c}})\mathbf{u}^t M_{\vec{c}}^{-1}\mathbf{u}. \tag{5.8}$$

For a fixed $\vec{c}$, this is a quadratic polynomial in the variable $\mathbf{u}$. For a fixed $\mathbf{u}$, this is a polynomial of degree $s-1$ in the projective variable $\vec{c}$ whose discriminant is given by the equation of the dual variety $\mathscr{F}^*(\mathbf{u})$.

From [16, Section 2], we know that rank $M_{\vec{c}} \geq s-1$ by the non-singular condition on $\vec{\mathrm{F}}$. A primitive integer pair $\vec{c}$ is called bad if the matrix $M_{\vec{c}}$ is singular, otherwise it is called good. Note that there are at most s pairs of bad $\vec{c}$. We divide the primes into the following categories:

(1) bad primes: $p \mid D_F$,
(2) good primes for bad $\vec{c}$: $p \nmid D_F$,
(3) good primes of Type I for good $\vec{c}$: $p \nmid D_F, p \nmid \det(M_{\vec{c}})$,
(4) good primes of Type II for good $\vec{c}$: $p \nmid D_F, p \mid \det(M_{\vec{c}})$.

When $\vec{c}$ is bad, the form $F_{\vec{c}}$ is singular and in particular $\rho_s = 0$. Similarly when p is a good prime of type II for good $\vec{c}$, we have $p \mid \det(M_{\vec{c}})$ and thus $p \mid \rho_s$. Let $\mathbf{y}_j = S^{-1}\mathbf{e}_j$ where $\{\mathbf{e}_j\}_{j=1}^s$ denotes the standard basis of $\mathbb{Z}^s$ and S is the invertible integer matrix appearing in the Smith normal form for $M_{\vec{c}}$. Note that $\{\mathbf{y}_i\}_{j=1}^s$ also forms a basis of $\mathbb{Z}^s$. Let $Q_{\vec{c}}$ denote the restriction of $F_{\vec{c}}$ to the $s-1$ dimensional subspace spanned by $\mathbf{y}_1, ..., \mathbf{y}_{s-1}$. Namely,

$$Q_{\vec{c}}(y_1, ..., y_{s-1}) := F_{\vec{c}}(y_1\mathbf{y}_1 + ... + y_{s-1}\mathbf{y}_{s-1}).$$

Similarly, $Q^*_{\vec{c}}$ denotes the dual form to the quadratic form $Q_{\vec{c}}$. Given $\mathbf{u} \in \mathbb{R}^s$, let $\mathbf{u}'$ denote the projection of $\mathbf{u}$ onto the $s-1$ dimensional subspace spanned by $\{\mathbf{y}_1, ..., \mathbf{y}_{s-1}\}$. Note that this is the hyperplane defined by the equation $\mathbf{y}_s \cdot \mathbf{x} = 0$. Moreover, since $\{\mathbf{y}_j\}_{j=1}^s$ forms a basis for $\mathbb{Z}^s$, if $\mathbf{u} \in \mathbb{Z}^s$, so is the projection $\mathbf{u}'$.

We remark that for bad $\vec{c}$ and good p, if we write $\Delta(Q_{\vec{c}})$ for the determinant of $Q_{\vec{c}}$, then we have $p \nmid \Delta(Q_{\vec{c}})$. Indeed if $p \mid \Delta(Q_{\vec{c}})$ then the rank of $M_{\vec{c}}$ over $\overline{\mathbb{F}_p}$ is $\leq s-2$, and so $F_1 = F_2 = 0$ is singular over $\overline{\mathbb{F}_p}$ and $D_F = 0$ over $\overline{\mathbb{F}}_p$ by [16, Proposition 2.1], hence $p \mid D_F$.

For a non-zero vector $\vec{\mathrm{w}}$, define

$$\lambda_{\vec{\mathrm{w}}} := \min_{1\le j\le s}\{|(\lambda_j,\mu_j)\cdot\vec{\mathrm{w}}|/|\vec{\mathrm{w}}|\}, \tag{5.9}$$

which measures the distance of the unit vector $\vec{\mathrm{w}}/|\vec{\mathrm{w}}|$ from the lines $\lambda_j x+\mu_j y=0$. Note $\lambda_{\vec{\mathrm{c}}}=0$ if and only if $\vec{\mathrm{c}}$ is bad.

Lemma 5.4. *Let $\vec{\mathrm{c}}\in\mathbb{Z}^2$ be a good, primitive integer vector, suppose $s\ge 4$and let $X\subseteq\mathbb{P}^{s-1}$ be the variety defined by $\mathscr{F}^*(\mathbf{u})=F^*_{\vec{\mathrm{c}}}(\mathbf{u})=0$. Then*

(i) X is a complete intersection of dimension $s-3$ defined over $\mathbb{Q}$.
(ii) X does not contain any linear space of projective dimension $\ge\lfloor\frac{s}{2}\rfloor$.
(iii) If $s\ge 7$ then X does not have any component of degree 3.

Proof. For part (i), it suffices to show that $F^*_{\vec{\mathrm{c}}}=0$ and $\mathscr{F}^*=0$ do not have a common irreducible component over $\mathbb{Q}$. But every component of $F^*_{\vec{\mathrm{c}}}=0$ has degree ≤ 2, while $\mathscr{F}^*=0$ is irreducible with degree $4(s-2)\ge 4$, so we are done.

For part (ii), observe that $\vec{\mathrm{c}}$ is good, so $F^*_{\vec{\mathrm{c}}}$ is a quadratic form of rank s. By Witt's Theorem, the maximum possible index of isotropy of $F^*_{\vec{\mathrm{c}}}$ is $\lfloor\frac{s}{2}\rfloor$, that is $F^*_{\vec{\mathrm{c}}}=0$ cannot contain a linear space of *projective* dimension $>\lfloor\frac{s}{2}\rfloor-1$ and *a fortiori* the same is true of X.

For (iii), suppose $V\subset\mathbb{P}^{s-1}$ is such a component. Then $\deg V=1+\operatorname{codim}V$, making V a *variety of minimal degree*. As V has degree 3 it is a cone over a rational normal scroll [12, Theorem 1]. In particular there are linear spaces $L\subseteq V$ with codimension 1 [12, p5]. Now by the previous part (ii), we have $\dim(V)-1<\lfloor\frac{s}{2}\rfloor$, while by part (i) we have $\dim(V)=s-3$. This implies $s<7$. ☐

Remark 5.5. The proof of (iii) is inspired by the mathoverflow answer of Sasha [34], which would actually permit us to weaken the hypothesis in (iii) to $s\ge 6$ by a closer inspection of the possibilities for V at the end of our argument. Presumably (ii) is not optimal either, but it is enough for our purposes.

6. Bounds for exponential integrals

Recall the exponential integral defined in (5.7). After a change of variables, we see that

$$I_q(\vec{\mathrm{w}},\mathbf{u})=P^s\int_{\mathbb{R}^s}w(\mathbf{x})\,e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)d\mathbf{x}.$$

We have the following pointwise bound for $I_q(\vec{\mathrm{w}},\mathbf{u})$.

Lemma 6.1. *If we let $\delta>0$, then for $|\mathbf{u}|\ge\frac{q}{P}(1+P^2|\vec{\mathrm{w}}|)P^\delta$, we have*

$$I_q(\vec{\mathrm{w}},\mathbf{u})\ll_N P^s(P|\mathbf{u}|/q)^{-N}\ll P^{s-N\delta}$$

for any $N>0$. Moreover for any $\mathbf{u}\in\mathbb{R}^s$,

$$|I_q(\vec{\mathrm{w}},\mathbf{u})|\ll P^s\prod_{j=1}^s(1+P^2|\lambda_j w_1+\mu_j w_2|)^{-1/2}\ll P^s(1+P^2|\vec{\mathrm{w}}|)^{-(s-1)/2}(1+P^2|\vec{\mathrm{w}}|\lambda_{\vec{\mathrm{w}}})^{-1/2},$$

where $\lambda_{\vec{\mathrm{w}}}$ is defined in (5.9).

Proof. The lemma follows from [16, Lemma 3.1, Lemma 2.3]. ☐

We also need some estimates of averaged bounds for $I_q(\vec{\mathrm{w}},\mathbf{u})$ integrated against $p_{1,q}$ and $p_{2,\vec{\mathrm{r}},k,q}$.

Lemma 6.2. *For any* $\mathbf{u} \in \mathbb{R}^s$ *we have*

$$\int_{|\vec{\mathrm{w}}| \asymp W} |p_{1,q}(\vec{\mathrm{w}})||I_q(\vec{\mathrm{w}}, \mathbf{u})| d\vec{\mathrm{w}} \ll \frac{Q}{q} W^2 P^s (1 + P^2 W)^{-s/2-1} (1 + |\vec{\mathrm{w}}| q Q^{1/2})^{-N}.$$

for any $N > 0$.

Proof. The lemma follows from [16, Lemma 3.3] together with Lemma 3.3 on the decay properties of $p_{1,q}$ and the relation $P^2 = Q^{3/2}$. □

Given $k \in \mathbb{N}$ and $\vec{\mathrm{r}} \in \mathbb{Z}^2$, we write for short

$$p_2(q, \vec{\mathrm{w}}) =: p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}}).$$

Lemma 6.3. *Let* $k \in \mathbb{N}$ *and* $\vec{\mathrm{r}} \in \mathbb{Z}^2$ *with* $|\vec{\mathrm{r}}| \asymp Q^{1/2}$ *and* $\vec{\mathrm{c}} = \vec{\mathrm{r}}/\gcd(\vec{\mathrm{r}})$. *Let* μ_{S^1} *be the Lebesgue measure (arc-length measure) on the unit circle* $S^1 \subset \mathbb{R}^2$. *For any* $W > 0$, *we have for* $j = 0, 1$

$$\begin{aligned}&\int_{|\vec{\mathrm{w}}| \asymp W} |q^j \frac{\partial^j}{\partial q^j} p_2(q, \vec{\mathrm{w}})||I_q(\vec{\mathrm{w}}, \mathbf{u})| d\vec{\mathrm{w}} \\ &\ll \mathbb{1}_{\frac{kq}{Q}<1} W Q^{-3/2} P^s (1 + kqQ^{1/2}W)^{-N} (1 + P^2 W)^{-(s-1)/2} (1 + P^2 W \lambda_{\vec{\mathrm{c}}})^{-1/2},\end{aligned}$$

and, where $\lambda_{\vec{\mathrm{c}}}$ *is as in* (5.9),

$$\begin{aligned}&W^2 \int_{S^1} |W^j \frac{\partial^j}{\partial W^j} p_2(q, W\vec{\mathrm{v}})||I_q(W\vec{\mathrm{v}}, \mathbf{u})| d\mu_{S^1}(\vec{\mathrm{v}}) \\ &\ll \mathbb{1}_{\frac{kq}{Q}<1} W Q^{-3/2} P^s (1 + kqQ^{1/2}W)^{-N} (1 + P^2 W)^{-(s-1)/2} (1 + P^2 W \lambda_{\vec{\mathrm{c}}})^{-1/2},\end{aligned}$$

the right-hand side being the same in both of these two estimates.

Proof. For $\vec{\mathrm{w}} = (w_1, w_2) \in \mathbb{R}^2$, we define its radial coordinates by $\vec{\mathrm{w}} = W\vec{\mathrm{v}}$ with $W \geq 0$ and $\vec{\mathrm{v}} \in S^1$.

We begin by considering the derivative $\frac{\partial}{\partial W} p_2(q, W\vec{\mathrm{v}})$ for $\vec{\mathrm{v}} \in S^1$. For fixed $\vec{\mathrm{v}}$, we have the identity $\frac{d}{dW} f(x(W), y(W)) = \frac{dx}{dW}\frac{\partial f}{\partial x} + \frac{dy}{dW}\frac{\partial f}{\partial y}$ with $x(W) = \frac{\vec{\mathrm{r}}\cdot\vec{\mathrm{w}}}{|\vec{\mathrm{r}}|}$, $y(W) = \frac{\vec{\mathrm{r}}^{\perp}\cdot\vec{\mathrm{w}}}{|\vec{\mathrm{r}}|}$ and $f(x, y) = p_{2,\vec{\mathrm{r}},k,q}(\vec{\mathrm{w}})$, which gives

$$\begin{aligned}\frac{\partial}{\partial W} p_2(q, \vec{\mathrm{w}}) &= \left(\frac{\vec{\mathrm{r}}}{|\vec{\mathrm{r}}|} \cdot \frac{dx(W)}{dW}\right) \partial_{\vec{\mathrm{r}}} p_2(q, \vec{\mathrm{w}}) + \left(\frac{\vec{\mathrm{r}}^{\perp}}{|\vec{\mathrm{r}}|} \cdot \frac{dy(W)}{dW}\right) \partial_{\vec{\mathrm{r}}^{\perp}} p_2(q, \vec{\mathrm{w}}) \\ &= \frac{\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}}{|\vec{\mathrm{r}}|} \partial_{\vec{\mathrm{r}}} p_2(q, \vec{\mathrm{w}}) + \frac{\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}^{\perp}}{|\vec{\mathrm{r}}|} \partial_{\vec{\mathrm{r}}^{\perp}} p_2(q, \vec{\mathrm{w}}),\end{aligned}$$

upon writing $\partial_{\vec{\xi}} = \frac{\xi_1}{|\vec{\xi}|}\frac{\partial}{\partial w_1} + \frac{\xi_2}{|\vec{\xi}|}\frac{\partial}{\partial w_2}$,. Together with Lemma 3.4, this gives

$$\begin{aligned}W \frac{\partial}{\partial W} p_2(q, \vec{\mathrm{w}}) &= O(W) \partial_{\vec{\mathrm{r}}} p_2(q, \vec{\mathrm{w}}) + O\left(\frac{|W\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}^{\perp}|}{Q^{1/2}}\right) \partial_{\vec{\mathrm{r}}^{\perp}} p_2(q, \vec{\mathrm{w}}) \\ &\ll (kqQ^{1/2}W + |W\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}^{\perp}|Q)(1 + kqQ^{1/2}W)^{-N}(1 + |W\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}^{\perp}|Q)^{-N} \\ &\ll \max\{1 + kqQ^{1/2}W, 1 + |W\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}^{\perp}|Q\}(1 + kqQ^{1/2}W)^{-N}(1 + |W\vec{\mathrm{v}} \cdot \vec{\mathrm{r}}^{\perp}|Q)^{-N} \\ &\ll (1 + kqQ^{1/2}W)^{-N+1}(1 + |W\vec{\mathrm{w}} \cdot \vec{\mathrm{r}}^{\perp}|Q)^{-N+1}, \qquad (6.1)\end{aligned}$$

for any $N > 0$.

Now let the differential operator D be defined either by $D = q\frac{\partial}{\partial q}$, or by $D = W\frac{\partial}{\partial W}$.

From Lemma 6.1 and from Lemma 3.4 for $D = q\frac{\partial}{\partial q}$ and (6.1) for $D = W\frac{\partial}{\partial W}$, we obtain the following estimate for $j = 0, 1$, $kq < Q$ and $\tau \in (1, 2]$:

$$\begin{aligned}&\int_{W\le|\vec{w}|\le\tau W} |D^j p_2(q,\vec{w})||I_q(\vec{w},\mathbf{u})|d\vec{w}\\ &\ll_N P^s \int_{W\le|\vec{w}|\le\tau W} (1+kqQ^{1/2}|\vec{w}|)^{-N}(1+Q\gcd(\vec{r})|\vec{c}^{\perp}\cdot\vec{w}|)^{-N}\prod_{j=1}^{s}(1+P^2|\lambda_i w_1+\mu_i w_2|)^{-1/2}d\vec{w}.\end{aligned}$$

We change variables to write $\vec{w} = x\frac{\vec{c}}{|\vec{c}|} + y\frac{\vec{c}^{\perp}}{|\vec{c}|}$ so that

$$\begin{aligned}&\int_{W\le|\vec{w}|\le\tau W} |D^j p_2(q,\vec{w})||I_q(\vec{w},\mathbf{u})|d\vec{w}\\ &\ll_N P^s \int_{W\le|\vec{w}|\le\tau W} (1+kqQ^{1/2}x)^{-N}(1+Q^{3/2}y)^{-N}\prod_{j=1}^{s}\left(1+P^2|(\lambda_j,\mu_j)\cdot(x\frac{\vec{c}}{|\vec{c}|}+y\frac{\vec{c}^{\perp}}{|\vec{c}|})|\right)^{-1/2}dxdy\\ &\ll_N P^sQ^{-3/2}\int_{W\le x\le\tau W}(1+kqQ^{1/2}x)^{-N}\prod_{j=1}^{s}(1+P^2|(\lambda_j,\mu_j)\cdot(x\vec{c}/|\vec{c}|)|)^{-1/2}dx\\ &\ll (\tau-1)WQ^{-3/2}P^s(1+kqQ^{1/2}W)^{-N}(1+P^2W)^{-(s-1)/2}(1+P^2W\lambda_{\vec{c}})^{-1/2},\end{aligned}$$

where we used [16, Lemma 2.3] to obtain the last inequality. Taking $D = q\frac{\partial}{\partial q}$, this concludes the first part of the lemma. Alternatively, dividing both sides by $(\tau - 1)$, letting τ go to 1, and taking $D = W\frac{\partial}{\partial W}$, we obtain the second part of the lemma. $\square$

When $|\mathbf{u}| \le \frac{q}{P}(1 + P^2W)P^{\varepsilon}$, we will also need a bound on average value of the derivative $\frac{\partial}{\partial q}I_q(\vec{w},\mathbf{u})$ integrated against $p_2(q,\vec{w})$ when we consider a double Kloosterman refinement.

Lemma 6.4. *Under the same assumptions as in Lemma 6.3, we have the following bound for any $N \ge 0$ and any interval $I \subseteq [1, 2]$*

$$\begin{aligned}&\int_{|\vec{w}|/W\in I} p_2(q,\vec{w})\frac{\partial}{\partial q}I_q(\vec{w},\mathbf{u})d\vec{w}\\ &\ll \mathbb{1}_{\frac{kq}{Q}<1}q^{-1}WQ^{-3/2}P^s(1+kqQ^{1/2}W)^{-N}(1+P^2W)^{-(s-1)/2}(1+P^2W\lambda_{\vec{c}})^{-1/2}.\end{aligned}$$

Proof. We note that

$$\frac{\partial}{\partial q}I_q(\vec{w},\mathbf{u}) = q^{-1}P^s\int_{\mathbb{R}^s}\frac{2\pi iP\mathbf{x}\cdot\mathbf{u}}{q}w(\mathbf{x})\,e(P^2\vec{w}\cdot\vec{F}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)d\mathbf{x}.$$

Treating the factor $\frac{P\mathbf{x}\cdot\mathbf{u}}{q}$ trivially will result in an extra factor of $1+P^2|\vec{w}|$ in case $\frac{P|\mathbf{u}\cdot\mathbf{x}|}{q} \ll P^{\varepsilon}(1+P^2|\vec{w}|)$, as Lemma 6.1 suffices otherwise. The goal is to remove the factor $P\mathbf{x}\cdot\mathbf{u}/q$ and we achieve this by integration by parts. We first write

$$
\begin{aligned}
q\frac{\partial}{\partial q}I_q(\vec{\mathrm{w}},\mathbf{u}) &= -P^s\int_{\mathbb{R}^s} w\left(\mathbf{x}\right)e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))\mathbf{x}\cdot\nabla e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\\
&= P^s\int_{\mathbb{R}^s}\mathrm{div}\left(w\left(\mathbf{x}\right)e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))\mathbf{x}\right)e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\\
&= -P^s\int_{\mathbb{R}^s}\left(\sum_{j=1}^{s}\frac{\partial}{\partial x_j}x_jw(\mathbf{x})\right)e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\\
&\quad+P^s\int_{\mathbb{R}^s}4\pi iP^2w(\mathbf{x})(\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x},
\end{aligned}
\tag{6.2}
$$

where we applied the divergence theorem in the second equality and used $\vec{\mathrm{F}}$ is homogeneous in the last equality so that $\sum_{j=1}^{s}x_jw(\mathbf{x})\frac{\partial}{\partial x_j}e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))=4\pi iP^2w(\mathbf{x})\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x})$.

The first term in (6.2) has the same form as I_q with the weight w replaced by $\sum_{j=1}^{s}\frac{\partial}{\partial x_j}x_jw(\mathbf{x})$, whose contribution in the integral against $p_2(q,\vec{\mathrm{w}})$ can be dealt with using first part of Lemma 6.3. To handle the second term in (6.2), we change to radial coordinates and then remove the factor $P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x})$ by integration by parts.

As in Lemma 6.3, let μ_{S^1} be the Lebesgue measure (arc-length measure) on the unit circle $S^1\subset\mathbb{R}^2$. Then, multiplying (6.2) by $p_2(q,\vec{\mathrm{w}})$ and integrating over $|\vec{\mathrm{w}}|\asymp W$, we must bound

$$
\begin{aligned}
&\int_{C_1W\le|\vec{\mathrm{w}}|\le C_2W}p_2(q,\vec{\mathrm{w}})P^s\int_{\mathbb{R}^s}4\pi iP^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x})w(\mathbf{x})e(P^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\,d\vec{\mathrm{w}}\\
&=\int_{S^1}\int_{C_1W}^{C_2W}tP^sp_2(q,t\vec{\mathrm{w}})\int_{\mathbb{R}^s}4\pi itP^2\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x})w(\mathbf{x})e(P^2t\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\,dt\,d\mu_{S^1}(\vec{\mathrm{w}}).
\end{aligned}
$$

Using integration by parts with respect to t, this is

$$
\begin{aligned}
&=2P^s\int_{S^1}\int_{\mathbb{R}^s}t^2p_2(q,t\vec{\mathrm{w}})w(\mathbf{x})e(P^2t\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\,\Big|_{C_1W}^{C_2W}d\mu_{S^1}(\vec{\mathrm{w}})\\
&\quad-4P^s\int_{S^1}\int_{C_1W}^{C_2W}\int_{\mathbb{R}^s}tp_2(q,t\vec{\mathrm{w}})w(\mathbf{x})e(P^2t\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\,dt\,d\mu_{S^1}(\vec{\mathrm{w}})\\
&\quad-2P^s\int_{S^1}\int_{C_1W}^{C_2W}t^2\int_{\mathbb{R}^s}\left(\frac{\partial}{\partial t}p_2(q,t\vec{\mathrm{w}})\right)w(\mathbf{x})e(P^2t\vec{\mathrm{w}}\cdot\vec{\mathrm{F}}(\mathbf{x}))e(-P\mathbf{x}\cdot\mathbf{u}/q)\,d\mathbf{x}\,dt\,d\mu_{S^1}(\vec{\mathrm{w}}).
\end{aligned}
$$

The second term is satisfactory by the first part of Lemma 6.3, and the other terms are satisfactory by the second part of Lemma 6.3. □

7. Bounds for quadratic exponential sums

In this section, we obtain bounds for two types of quadratic exponential sums: the sums $D_q(\mathbf{u})$ in (5.5) and $S_{q,d\vec{c}}(\mathbf{u})$ in (5.6). The exponential sums $D_q(\mathbf{u})$ are closely related to $S_{d,1}(\mathbf{u})$, defined in [6, eq. (1.2)]. The exponential sums $S_{q,d\vec{c}}(\mathbf{u})$ are closely related to the exponential sums studied in [35] in the function field setting. We will adapt the methods in [6] and [35] to study these sums in our setting.

Using Chinese remainder theorem, we have the following multiplicativity properties for D_q and $S_{q,d\vec{c}}$.

Lemma 7.1. *Let $s, q_1, q_2, d_1, d_2 \geq 1$ be integers satisfying $\gcd(q_1, q_2) = 1$ and $d_1 \mid q_1, d_2 \mid q_2$. Then for any primitive integer vector $\vec{c} \in \mathbb{Z}^2$, and any integer vector $\mathbf{u} \in \mathbb{Z}^s$, we have the following multiplicativity relations:*

$$D_{q_1 q_2}(\mathbf{u}) = D_{q_1}(\mathbf{u}) D_{q_2}(\mathbf{u}),$$
$$S_{q_1 q_2, d_1 d_2 \vec{c}} = S_{q_1, d_1 \vec{c}}(\mathbf{u}) S_{q_2, d_2 \vec{c}}(\mathbf{u}).$$

7.1. **Bounds for $D_q(\mathbf{u})$.** In this section, we give estimates for $D_q(\mathbf{u})$. By multiplicativity, it is enough to consider the case when $q = p^k$ where p is prime and $k \in \mathbb{N}$.

Lemma 7.2. *For $\mathbf{u} \in \mathbb{Z}^s$ we have*

$$D_p(\mathbf{u}) \ll \begin{cases} p^{(s+2)/2} & p \nmid \mathscr{F}^*(\mathbf{u}), \\ p^{(s+3)/2} & p \mid \mathscr{F}^*(\mathbf{u}), p \nmid \mathbf{u}. \end{cases}$$

Here, $\mathscr{F}^*(\mathbf{u})$ denotes the dual variety for the intersection of quadrics F_1 and F_2 defined in Section 5.2.

Proof. We write $D_p(\mathbf{u}) = \sum_{\vec{a} \bmod p} \sum_{\mathbf{b} \bmod p} e_p(\vec{a} \cdot \vec{F}(\mathbf{b}) + \mathbf{b} \cdot \mathbf{u}) - \sum_{\mathbf{b} \bmod p} e_p(\mathbf{b} \cdot \mathbf{u})$. The lemma follows from [6, Lemma 19] since the second sum vanishes unless $p \mid \mathbf{u}$. □

Lemma 7.3. *Let $k \geq 2, \mathbf{u} \in \mathbb{Z}^s$, we have $D_{p^k}(\mathbf{u}) = 0$ unless $p \mid D_F \mathscr{F}^*(\mathbf{u})$.*

Proof. We write

$$D_{p^k}(\mathbf{u}) = \sum_{\vec{a} \bmod p^k} \sum_{\mathbf{b} \bmod p} e_{p^k}(\vec{a} \cdot \vec{F}(\mathbf{b}) + \mathbf{b} \cdot \mathbf{u}) - p^s \mathbb{1}_{p|\mathbf{u}} \Big(\sum_{\vec{a} \bmod p^{k-1}} \sum_{\mathbf{b} \bmod p^{k-1}} e_{p^{k-1}}(\vec{a} \cdot \vec{F}(\mathbf{b}) + \mathbf{b} \cdot \frac{\mathbf{u}}{p}) \Big).$$

If $p \nmid D_F \mathscr{F}^*(\mathbf{u})$, then $p \nmid \mathbf{u}$ and the lemma then follows from an application of Hensel's lemma akin to [6, Lemma 20]. □

Lemma 7.4. *For any $q \geq 1$ and $\mathbf{u} \in \mathbb{Z}^s$, we have $D_q(\mathbf{u}) \ll q^{s/2+2+\varepsilon}$ for any $\varepsilon > 0$.*

Proof. By multiplicativity, it is enough to check the bound for $q = p^k$ for some integer k. After a standard squaring argument such as [35, Lemma 2.5],

$$|\sum_{\mathbf{b} \bmod q} e_q(\vec{a} \cdot \vec{F}(\mathbf{b}) + \mathbf{b} \cdot \mathbf{u})| \ll q^{s/2} Z(\vec{a}; q)^{1/2},$$

where $Z(\vec{a}; q) = \#\{\mathbf{z} \bmod q : q \mid \mathbf{z}^t(a_1 M_1 + a_2 M_2), \}$ where M_1 and M_2 are matrices defining the quadratic forms. As a result,

$$|D_{p^r}(\mathbf{u})| \ll p^{ks/2} \sideset{}{^*}\sum_{\vec{a} \bmod p^k} Z(\vec{a}; p^k)^{1/2} \ll p^{ks/2+k} \Big(\sideset{}{^*}\sum_{\vec{a} \bmod p^k} Z(\vec{a}; p^k) \Big)^{1/2} \ll p^{ks/2+2k+\varepsilon} = q^{s/2+2+\varepsilon},$$

for any $\varepsilon > 0$ upon using [16, Lemma 5.4]. □

Combining Lemmas 7.2–7.4, we obtain the following upper bound for $D_q(\mathbf{u})$.

Lemma 7.5. *Suppose $q = q_1 q_2$ where q_1 is the square-free part of q and q_2 is the square-full part of q. Then for any $\varepsilon > 0$*

$$D_q(\mathbf{u}) \ll q^{s/2+1+\varepsilon} \gcd(q_1, \mathscr{F}^*(\mathbf{u}))^{1/2} \gcd(q_1, \mathbf{u})^{1/2} \gcd(q_2, (D_F \mathscr{F}^*(\mathbf{u}))^\infty).$$

7.2. **Bounds for $S_{q,d\vec{c}}(\mathbf{u})$.** Recall the definition of $S_{q,d\vec{c}}(\mathbf{u})$ in (5.6). The function field analogue of similar exponential sums defined in [35, eq. (6.1)] have been extensively studied there. When $(d,q)=1$, the exponential sum $S_{q,\vec{c}}(\mathbf{u})$ coincides with that considered in [35, Section 6] in the function field setting and analogous estimates can be proved similarly. By multiplicativity, it is enough to consider sums $S_{p^k,\vec{c}}(\mathbf{u})$. In the next four lemmas, we will obtain estimates for $S_{p^k,\vec{c}}$ when p is a good.

Lemma 7.6 (Type I primes for good $\vec{c}$)**.** *Let $\vec{c}$ be good. For $p\nmid 2\det M_{\vec{c}}$, we have*

$$|S_{p^k,\vec{c}}(\mathbf{u})|\le\begin{cases}p^{sk/2}\gcd(F^*_{\vec{c}}(\mathbf{u}),p^k) & 2\mid s \text{ or } 2\nmid s, 2\mid k,\\ p^{k(s+1)/2}\gcd(F^*_{\vec{c}}(\mathbf{u}),p^k)^{1/2}\mathbb{1}_{F^*_{\vec{c}}(\mathbf{u})\neq 0} & 2\nmid s, 2\nmid k.\end{cases}$$

Proof. Since $\vec{c}$ is primitive, we see that $p^k\mid\vec{c}\cdot\vec{a}^{\perp}$ in (5.6) implies that $\vec{a}=a\vec{c}$ for some $(a,p)=1$ so that (5.6) can be written as

$$S_{p^k,\vec{c}}(\mathbf{u})=\sideset{}{^*}\sum_{a \bmod p^k}\sum_{\mathbf{b}\bmod p^k}e_{p^k}(a\vec{c}\cdot\vec{F}(\mathbf{b})+\mathbf{b}\cdot\mathbf{u})=\sideset{}{^*}\sum_{a \bmod p^k}\sum_{\mathbf{b}\bmod p^k}e_{p^k}(aF_{\vec{c}}(\mathbf{b})+\mathbf{b}\cdot\mathbf{u}).$$

From [6, Lemma 15], we have for $p\nmid 2\det M_{\vec{c}}$

$$S_{p^k,\vec{c}}(\mathbf{u})=\begin{cases}p^{sk/2}\varepsilon(p)^{sk}\chi_p(\det M_{\vec{c}})^k c_{p^k}(F^*_{\vec{c}}(\mathbf{u})) & 2\mid s,\\ p^{sk/2}c_{p^k}(F^*_{\vec{c}}(\mathbf{u})) & 2\nmid s, 2\mid k,\\ p^{sk/2}\varepsilon(p)^s\chi_p(-1)g_{p^k}(F^*_{\vec{c}}(\mathbf{u})) & 2\nmid s,2\nmid k,\end{cases}\tag{7.1}$$

where $\chi_p(\cdot)=(\frac{\cdot}{p})$ is the Legendre Symbol, $\varepsilon(p)=\begin{cases}1 & p\equiv 1\bmod 4\\ i & p\equiv 3 \bmod 4\end{cases}$, $F^*_{\vec{c}}$ defined in (5.8) is the adjoint quadratic form of $F_{\vec{c}}$, $c_{p^k}(a)=\sum^*_{x\bmod p^k}e_{p^k}(ax)$ is the Ramanujan sum, and $g_{p^k}(a)=\sum_{x\bmod p^k}\chi_p(x)e_{p^k}(ax)$ is the Gauss sum. The lemma follows from the bounds $|c_{p^k}(a)|\le(a,p^k)$ and $|g_{p^k}(a)|\le p^{k/2}(a,p^k)^{1/2}\mathbb{1}_{a\neq 0}$. □

While dealing with a pair of quadrics over $s=9$ variables, we will need to make use of cancellations over averages of the exponential sums $S_{q,\vec{c}}(\mathbf{u})$ over q, which is the content of the following lemma.

Lemma 7.7. *Let s be an odd integer, $\vec{c}\in\mathbb{Z}^2$ be a good pair, and let $\mathbf{u}\in\mathbb{Z}^s$ be such that $F^*_{\vec{c}}(\mathbf{u})$ is not a perfect square. Assume the generalized Lindelöf Hypothesis for Dirichlet L-functions. Then for any integer $\mathfrak{B}$ such that $2\det M_{\vec{c}}\mid\mathfrak{B}$ and any $\varepsilon>0$, we have that*

$$\Big|\sum_{\substack{1\le q\le x\\ \gcd(q,\mathfrak{B})=1}}S_{q,\vec{c}}(\mathbf{u})\Big|\ll(\mathfrak{B}|\vec{c}||\mathbf{u}|)^{\varepsilon}x^{s/2+1+\varepsilon}.$$

Proof. This can be viewed as a conditional version of [6, Lemma 18] and can be proved in the same way. Let

$$\xi_{\mathfrak{B}}(z,\mathbf{u})=\prod_{p\nmid\mathfrak{B}}\Big(\varepsilon(p)^s\chi_p(-1)\sum_{2\nmid k}\frac{g_{p^k}(F^*_{\vec{c}}(\mathbf{u}))}{p^{k(z-s/2)}}+\sum_{2\mid k}\frac{c_{p^k}(F^*_{\vec{c}}(\mathbf{u}))}{p^{k(z-s/2)}}\Big).$$

Since $g_p(a)=\chi_p(a)\varepsilon(p)p^{1/2}$, we see that

$$\xi_{\mathfrak{B}}(z,\mathbf{u})=L(z-\frac{s+1}{2},\psi_{\mathbf{u}})E_{\mathfrak{B}}(z),$$

where $\psi_{\mathbf{u}}(\cdot) = \left(\frac{(-1)^{\frac{s+1}{2}} F^*_{\vec{c}}(\mathbf{u})}{\cdot}\right)$ is a Dirichlet character with conductor $O(|\vec{c}|^{s-1}|\mathbf{u}|^2)$ and $E_{\mathfrak{B}}(z)$ is an Euler product, which is absolutely convergent when $\Re(z) > s/2+1$ and satisfies the bound $E_{\mathfrak{B}}(z) \ll \mathfrak{B}^{\varepsilon}$, for any $\varepsilon > 0$. Thus $\xi_{\mathfrak{B}}(z, \mathbf{u})$ can be analytic continued to the region $\Re(z) > s/2+1$. Using Perron's formula and (7.1), we have for $c > s/2+2$

$$\sum_{\substack{1 \le q \le x \\ (q,\mathfrak{B})=1}} S_{q,\vec{c}}(\mathbf{u}) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \xi_{\mathfrak{B}}(z, \mathbf{u}) \frac{x^z}{z} dz + O(\frac{x^c}{T}).$$

Moving the line of integration to $s/2+1+\varepsilon$, using the bound $\xi_{\mathfrak{B}}(z, \mathbf{u}) \ll (|\vec{c}|^{s-1}|\mathbf{u}|^2 \mathfrak{B} T)^{\varepsilon}$, and taking T to be a sufficiently large power of x, we obtain the result. □

Lemma 7.8 (Type II primes for good $\vec{c}$)**.** *Let $\vec{c}$ be good. Let $M_{\vec{c}} = TDS$ where $T, S \in GL_s(\mathbb{Z})$ and $D = diag(\rho_i)$ with $\rho_i \mid \rho_{i+1}$. Suppose $p \nmid D_F$ and $p \mid \det(M_{\vec{c}})$ and let $k_1 = \min(k, \nu_p(\rho_s))$, where for any integer n, $\nu_p(n)$ is the largest integer k such that $p^k \mid n$. Then $S_{p^k}(\vec{c}, \mathbf{u})$ is non-zero unless $p^{k_1} \mid ((S^{-1})^T \mathbf{u})_s$, and we also have*

$$|S_{p^k,\vec{c}}(\mathbf{u})| \ll p^{k(s/2+1)} \gcd(p^{k_1}, Q^*_{\vec{c}}(\mathbf{u}'), ((S^{-1})^T \mathbf{u})_s)^{1/2},$$

where $\mathbf{u}'$ is the $s-1$ dimensional vector projection onto the hyperplane $\mathbf{y}_s \cdot \mathbf{x} = 0$.

Proof. This can be proved in the same way as the function field analogue [35, Lemma 6.3]. □

We now move on to bad pairs $\vec{c}$. In this case, the matrix $M_{\vec{c}}$ is singular and therefore the last diagonal entry $\rho_s = 0$ in the Smith normal decomposition of $M_{\vec{c}}$ as stated before. We also have

$$F_{\vec{c}}(\mathbf{x}) = Q_{\vec{c}}(\mathbf{x}'),$$

where as before $\mathbf{x}'$ denotes the projection of $\mathbf{x}$ onto the hyperplane $\mathbf{y}_s \cdot \mathbf{x} = 0$.

Lemma 7.9 (Good primes for bad $\vec{c}$)**.** *Let $\vec{c}$ be bad. Then for a good prime p, we have $S_{p^k,\vec{c}}(\mathbf{u})$ vanishes unless $p^k \mid ((S^{-1})^T \mathbf{u})_s$, and we have*

$$|S_{p^k,\vec{c}}(\mathbf{u})| \ll p^k \delta_{p^k | ((S^{-1})^t \mathbf{u})_s} \begin{cases} p^{k(s/2-1/2)} \gcd(p^k, Q^*_{\vec{c}}(\mathbf{u}')) & \text{if } 2 \mid k \text{ or } 2 \nmid k, 2 \nmid s, \\ p^{ks/2} & \text{if } 2 \nmid k, 2 \mid s. \end{cases}$$

Proof. This can be proved in the same way as the function field analogue [35, Lemma 6.4]. □

Next, we consider general bounds for $S_{q,d\vec{c}}$ with $(q, d) > 1$. When $(d, q) > 1$, the definition of $S_{q,d\vec{c}}(\mathbf{u})$ differ slightly from that in [35, Section 6] due to the co-primality condition in $L(d\vec{c})$ defined in [35], but the same bounds apply by following the proofs in [35, Lemma 6.5, Lemma 6.7]. By multiplicativity, it is again enough to consider $S_{p^k,p^m\vec{c}}$ with $1 \le m \le k$. We start with the following lemma corresponding to [35, Lemma 6.7], which will be used to obtain a general bound for $S_{q,d\vec{c}}$.

Lemma 7.10. *Let $\vec{c} \in \mathbb{Z}^2$ be primitive. For integers $1 \le m \le k$, denote $k_1 = \min(k-m, \nu_p(\det M_{\vec{c}}))$. Then we have*

$$|S_{p^k,p^m\vec{c}}(\mathbf{u})| \ll_{D_F} p^{k(n/2+1)+m+k_1/2} \mathbb{1}_{p^{k_1} | ((S^{-1})^T \mathbf{u})_s}.$$

Proof. Akin to [35, Lemma 3.3], we see that

$$\{\vec{a} \bmod p^k : (\vec{a}, p) = 1, p^k \mid p^m \vec{c} \cdot \vec{a}\} = \{a\vec{c}^{\perp} + p^{k-m}\vec{d} \bmod p^k : (a, p) = 1, a \bmod p^{k-m}, \vec{d} \bmod p^m\},$$

so that we can write

$$\sum_{\substack{\vec{a} \bmod p^k \\ (\vec{a},p)=1 \\ p^{k-m}|\vec{a}\cdot\vec{c}}} \sum_{\mathbf{b} \bmod p^k} e_{p^k}(\vec{a}\cdot F(\mathbf{b}) + \mathbf{b}\cdot\mathbf{u}) = \sum_{\vec{d} \bmod p^m} \sum_{\substack{a \bmod p^{k-m} \\ (a,p)=1}} e_{p^k}((a\vec{c}^{\perp} + p^{k-m}\vec{d})\cdot F(\mathbf{b}) + \mathbf{b}\cdot\mathbf{u}).$$

Then the proof follows the same way as [35, Lemma 6.7], using that uniformly for $(a,p)=1$, the following holds:

$$\sum_{\vec{d} \bmod p^m} \gcd(F(a\vec{c}^{\perp} + p^{k-m}\vec{d}), p^k)^{1/2} \ll p^{2m+k_1/2},$$

since $\vec{c}$ is primitive so that $(a\vec{c}^{\perp} + p^{k-m}\vec{d}, p) = 1$ for all $(a,p)=1$ and $\vec{d}\in\mathbb{Z}^2$. □

As a corollary of Lemma 7.10, we obtain the following analogue of [35, Lemma 6.8].

Lemma 7.11 (general bound)**.** *For $d \mid q$, we have*

$$\begin{aligned} |S_{q,d\vec{c}}(\mathbf{u})| &\ll_{D_F} dq^{s/2+1}\gcd(q/d, ((S^{-1})^T\mathbf{u})_s, \det M_{\vec{c}})^{1/2} \\ &\ll_{\vec{F}} d^{1/2}q^{s/2+3/2}. \end{aligned}$$

To estimate the contribution from bad pairs $\vec{c}$, we need the following refined estimate, that saves a factor of order $O(d^{1/2})$ as compared to the bound in Lemma 7.11 for square-free moduli, analogous to [35, Lemma 6.5].

Lemma 7.12. *Let $\vec{c}$ be a bad pair. Then*

$$|S_{p,p\vec{c}}(\mathbf{u})| \ll p^{s/2+1}(p,\mathbf{u})^{1/2}\gcd(p, \mathscr{F}^*(\mathbf{u}))^{1/2}.$$

Proof. The Lemma follows by noting that $S_{p,p\vec{c}}(\mathbf{u}) = D_p(\mathbf{u})$ and Lemma 7.5. □

8. Major arcs contribution: the main term

We obtain an asymptotic formula for the main contribution $N_0(P,\delta)$ in Lemma 5.3 in this section.

First we show that only $\mathbf{u}=\mathbf{0}$ contribute to the main term in $N_0(P,\delta)$.

Lemma 8.1. *For any $\delta, N > 0$, we have*

$$N_0(P,\delta) = \sum_{1\le q<Q^{1/2-\delta}} q^{-s}D_q(\mathbf{0})\int_{\vec{w}\in\mathfrak{M}_q(\delta)} I_q(\vec{w},\mathbf{0})\,d\vec{w} + O_N(P^{-N}).$$

Proof. Note that for $Q=P^{4/3}$, $q\le Q^{1/2-\delta}$, and any $\vec{w}\in\mathfrak{M}_q(\delta)$, we have

$$\frac{q}{P}(1+P^2|\vec{w}|) \ll \frac{q}{P}(1+P^2\frac{1}{qQ^{1+\delta}}) \ll \frac{q}{P}+\frac{P}{Q} \ll P^{-1/3}.$$

Thus from the first part of Lemma 6.1, we conclude that the contributions from all non-zero $\mathbf{u}$ are negligible in $N_0(P,\delta)$. □

To further evaluate $N_0(P,\delta)$, we define the singular series

$$\mathfrak{S} := \sum_{q=1}^{\infty} q^{-s} D_q(\mathbf{0}) \tag{8.1}$$

and the singular integral $\mathfrak{J}$

$$\mathfrak{J} := \int J(\vec{\mathrm{w}}) d\vec{\mathrm{w}}, \text{ where } J(\vec{\mathrm{w}}) = \int w(\mathbf{x}) e(\vec{\mathrm{w}} \cdot \vec{\mathrm{F}}(\mathbf{x})) d\mathbf{x}. \tag{8.2}$$

Lemma 8.2. *The singular integral $\mathfrak{J}$ is absolutely convergent as soon as $s \geq 6$ and the singular series $\mathfrak{S}$ is absolutely convergent for any $s \geq 7$. Moreover, for any $s \geq 7$ and any $\varepsilon > 0$, we have*

$$N_0(P,\delta) = P^{s-4} \left(\mathfrak{S}\mathfrak{J} + O(P^{-(s-6)/3+\varepsilon}) \right) \tag{8.3}$$

for sufficiently small δ (depending on ε and s).

Proof. First we show the convergence of $\mathfrak{J}$ and $\mathfrak{S}$. Using Lemma 6.1, we see that

$$J(\vec{\mathrm{w}}) \ll (1 + |\vec{\mathrm{w}}|)^{-(s-1)/2},$$

and it follows that

$$\int_{|\vec{\mathrm{w}}| \gg W} J(\vec{\mathrm{w}}) d\vec{\mathrm{w}} \ll \int_{|\vec{\mathrm{w}}| \gg W} (1 + |\vec{\mathrm{w}}|)^{-(s-1)/2} d\vec{\mathrm{w}} \ll \int_{r \gg W} r^{-(s-1)/2+1} dr \ll W^{-s/2+5/2}. \tag{8.4}$$

The singular integral is therefore absolutely convergent for $s \geq 6$. From Lemma 7.5, we have the bound

$$|D_q(\mathbf{0})| \ll q^{s/2+2+\varepsilon}, \tag{8.5}$$

for any $\varepsilon > 0$, which implies that

$$\sum_{q \geq 1} q^{-s} |D_q(\mathbf{0})| \ll \sum_{q \geq 1} q^{-s/2+2+\varepsilon}$$

for any $\varepsilon > 0$. Therefore, the singular series $\mathfrak{S}$ converges absolutely for all $s \geq 7$.

Next we evaluate $N_0(P,\delta)$. Using Lemma 8.1 and a change of variables,

$$\int_{|\vec{\mathrm{w}}| < q^{-1} Q^{-1-\varepsilon}} I_q(\vec{\mathrm{w}}, \mathbf{0}) = P^{s-4} \int_{|\vec{\mathrm{w}}| < q^{-1} Q^{1/2-\delta}} J(\vec{\mathrm{w}}) d\vec{\mathrm{w}},$$

we see that

$$N_0(P,\delta) = P^{s-4} \sum_{1 \leq q < Q^{1/2-\delta}} q^{-s} D_q(\mathbf{0}) \int_{|\vec{\mathrm{w}}| < q^{-1} Q^{1/2-\delta}} J(\vec{\mathrm{w}})\, d\vec{\mathrm{w}} + O_N(P^{-N}). \tag{8.6}$$

Using (8.4), we can we can replace the integral over $\vec{\mathrm{w}}$ by $\mathfrak{J}$ with an error

$$O\big((q^{-1} Q^{1/2-\delta})^{-s/2+5/2}\big) = O(q^{(s-5)/2} Q^{-1+\varepsilon}),$$

for any $s \geq 7$, provided δ is sufficiently small depending on ε and s.

Using (8.5), it then follows that for $s \geq 7$ and for sufficiently small δ we have

$$\begin{aligned}
&\sum_{1\leq q\leq Q^{1/2-\delta}} q^{-s} D_q(\mathbf{0}) \int_{|\vec{\mathrm{w}}|<q^{-1}Q^{1/2-\delta}} J(\vec{\mathrm{w}})\, d\vec{\mathrm{w}} \\
&= \mathfrak{J} \sum_{1\leq q< Q^{1/2-\delta}} q^{-s} D_q(\mathbf{0}) + O\Big(\sum_{1\leq q<Q^{1/2-\delta}} q^{-s/2+2} q^{(s-5)/2} Q^{-1+\varepsilon}\Big) \\
&= \mathfrak{J}\mathfrak{S} + O\Big(\sum_{q\geq Q^{1/2-\delta}} q^{-s/2+2+\varepsilon} + Q^{-1+\varepsilon} \sum_{q\leq Q^{1/2-\delta}} q^{-1/2}\Big) \\
&= \mathfrak{J}\mathfrak{S} + O(Q^{-(s-6)/4+\varepsilon} + Q^{-3/4+\varepsilon})) = \mathfrak{J}\mathfrak{S} + O(P^{-(s-6)/3+\varepsilon}),
\end{aligned}$$

which together with (8.6) yields (8.3). □

9. Minor arcs contribution: preparation

The process of proving the estimates for $N_1(P,\delta)$ and $N_2(P,\delta)$ from the minor arcs in Lemma 5.3 is similar to that in [35]. However, the situation at hand is more complicated due to the different weight functions $p_{1,q}, p_{2,\vec{\mathrm{r}},k,q}$ appearing in $N_1(P,\delta), N_2(P,\delta)$ in (5.3)-(5.4). Our bounds for the p-functions get increasingly worse as the sizes of q and $|\vec{\mathrm{w}}|$ decrease. However, in the extreme case

$$q \asymp Q,\ \ |\vec{\mathrm{w}}| \asymp q^{-1} Q^{-1/2+\delta},$$

our bounds match those in [35].

In this section, we prepare two lemmas to estimate the contributions from the $\mathbf{u}$-sum and $\vec{\mathrm{c}}$-sum. Our first estimate is essentially the uniform version of the dimension growth theorem of Salberger [32, Theorem 0.3], however we will use an affine version.

Lemma 9.1 (Dimension growth). *Let $X \subset \mathbb{P}^{s-1}$ be an irreducible variety over $\mathbb{Q}$ satisfying $\dim X \geq 1$, $\deg X \geq 2$ and $s \geq 3$. Let*

$$\eta(d) = \begin{cases} 0 & (d \neq 3), \\ \frac{2}{\sqrt{3}} - 1 & (d = 3). \end{cases}$$

Let $\mathscr{Y} \subset \mathbb{A}^s_{\mathbb{Z}}$ be an integral model of the affine cone over X. Then for $V \geq 1$ and any $\varepsilon > 0$

$$\{\mathbf{x} \in \mathscr{Y}(\mathbb{Q}) \cap \mathbb{Z}^s : |\mathbf{x}| \leq V\} \ll_{\deg X, s, \varepsilon} V^{\dim X + \eta(\deg X) + \varepsilon}.$$

Proof. Let $d = \deg X$.

When $d \geq 3$, Salberger proved [32, Theorem 0.3] that

$$\{\mathbf{x} \in \mathscr{Y}(\mathbb{Q}) \cap \mathbb{Z}^s : \gcd(\mathbf{x}) = 1, |\mathbf{x}| \leq V\} \ll_{d,s,\varepsilon} V^{\dim X + \eta(d) + \varepsilon}. \tag{9.1}$$

Salberger states this for integral varieties; recall that for us all varieties are reduced, and thus integral and irreducible are interchangeable terms.

We prove (9.1) in the case $d = 2$. Suppose first that X is geometrically reducible. In that case X is a union of two hyperplanes conjugate over a quadratic extension of $\mathbb{Q}$. All the points in $X(\mathbb{Q})$ lie on the intersection of these hyperplanes, which is a projective linear space of dimension $\dim X - 1$; thus $\mathscr{Y}(\mathbb{Q})$ is contained in an affine linear space of dimension $\dim X$ and therefore makes an acceptable contribution to the counting function in the lemma. In the remaining case when X is geometrically irreducible, (9.1) is proved in Heath-Brown [15, Theorem 2].

We now need to remove the gcd condition in the above estimate. Clearly, using (9.1),

$$\{\mathbf{x} \in \mathscr{Y}(\mathbb{Q}) \cap \mathbb{Z}^s : |\mathbf{x}| \leq V\} \ll \sum_{1 \leq m \leq V} (V/m)^{\dim X + \varepsilon} \ll V^{\dim X + \eta(\deg X) + \varepsilon},$$

as soon as $\dim X \geq 1$. □

We will also need the following counting lemma generalizing [35, Lemma 8.1].

Lemma 9.2 (Counting lemma)**.** *For any $s \geq 2$, $q, d \in \mathbb{N}$ with $d \mid q$, any primitive $\vec{c} \in \mathbb{Z}^2$ and any $V \geq 1$, we have that*

$$\#\{|\mathbf{u}| \leq V : q \mid ((S^{-1})^T \mathbf{u})_s, d \mid Q^*_{\vec{c}}(\mathbf{u}')\} \ll V^{s-2} \min\Big(V(1 + \frac{V}{q}), (1 + \frac{V}{\prod_{p|d} p})(1 + \frac{V}{\prod_{p|q} p})\Big), \tag{9.2}$$

and for $N \geq 0$, $x \in \mathbb{N}$ and any $\varepsilon > 0$,

$$\sum_{\substack{\vec{c} \text{ good} \\ |\vec{c}| \asymp C \\ x | \det M_{\vec{c}}}} (1 + N\lambda_{\vec{c}})^{-1/2} \ll_{F,\varepsilon} x^\varepsilon C\Big(1 + \frac{C}{x^{1/2}(1+N)^{1/2}}\Big), \tag{9.3}$$

where $\lambda_{\vec{c}}$ is as defined in (5.9).

Proof. Under the assumptions of the lemma, equation (9.2) can be proved the same way as its function field analogue in [35, eq. (8.6)] and we omit the details here.

Equation (9.3) is a generalization of [35, eq. (8.7)]. We show that for any $x_1 \mid x$ satisfying $x_1 \geq x^{1/2}$

$$\sum_{\substack{\vec{c} \text{ good} \\ |\vec{c}| \asymp C \\ x | \det M_{\vec{c}} \\ \gcd(x_1, c_1) = 1}} (1 + N\lambda_{\vec{c}})^{-1/2} \ll_{F,\varepsilon} x^\varepsilon C\Big(1 + \frac{C}{x_1(1+N)^{1/2}}\Big), \tag{9.4}$$

which implies (9.3).

Fix a value of c_1 co-prime to x_1. Following the proof of [35, eq. (8.7)], one can show that there exist integers $0 \leq b_1, ..., b_K < x_1$ where $K = O_{D_F}(x_1^\varepsilon)$ such that every $|c_2| \ll C$ satisfying $\gcd(c_1, c_2) = 1$ and $x_1 \mid \det(M_{\vec{c}})$ must be of the form $b_i + kx_1$ for some $|k| \ll 1 + C/x_1$. Therefore

$$\sum_{\substack{\vec{c} \text{ good} \\ |\vec{c}| \asymp C \\ x | \det M_{\vec{c}} \\ \gcd(c_1, x_1) = 1}} (1 + N\lambda_{\vec{c}})^{-1/2} \ll \sum_{\substack{|c_1| \ll C \\ \gcd(c_1, x) = 1}} \sum_{j=1}^{K} \sum_{|k| \ll 1 + C/x_1} (1 + N\lambda_{(c_1, b_j + kx_1)})^{-1/2}.$$

Now, we have uniformly for any $0 \leq b < x_1$ and c_1

$$\begin{aligned}
\sum_{|k| \ll C/x_1} (1 + N\lambda_{(c_1, b + kx_1)})^{-1/2} &\ll 1 + \sum_{0 \neq |k| \ll C/x_1} (1 + Nkx_1/C)^{-1/2} \\
&\ll 1 + \sum_{0 \neq |k| \leq \frac{C}{x_1(1+N)}} 1 + \sum_{\frac{C}{x_1(1+N)} \ll |k| \ll \frac{C}{x_1}} C^{1/2} N^{-1/2} x_1^{-1/2} k^{-1/2} \\
&\ll 1 + \frac{C}{x_1(1+N)^{1/2}},
\end{aligned}$$

which is enough to establish (9.4), upon recalling that $K = O_{D_F}(x^{\varepsilon})$. □

10. The Error Term $N_1(P,\delta)$

The goal of this section is to show that $N_1(P,\delta)$ in (5.3) satisfies the bound in Lemmam 5.3: for any $\varepsilon > 0$

$$N_1(P,\delta) \ll P^{s-4-(s-8)/3+\varepsilon} \tag{10.1}$$

when δ is sufficiently small depending on ε and s.

From Lemmas 5.2 and 6.1, we see that

$$N_1(P,\delta) = \sum_{1\le q\le Q} q^{-s} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\le \frac{q}{P}(1+P^2|\vec{w}|)P^{\varepsilon}}} D_q(\mathbf{u}) \int_{\mathfrak{m}_q(\delta)} p_{1,q}(\vec{w}) I_q(\vec{w},\mathbf{u})d\vec{w} + O_{N,\varepsilon}(P^{-N}), \tag{10.2}$$

for any $\varepsilon, N > 0$. Although we write $O_{N,\varepsilon}$ in the line above for clarity, we recall before taking the next step that our implicit constants can always depend on s, w, F_1, F_2 and ε, and that ε can vary between occurrences. Moreover we split $q, \vec{w}$ into dyadic ranges $q \asymp Y, |\vec{w}| \asymp W$, we see that $\mathbf{u}$ can be truncated up to V, where

$$V = \frac{Y}{P}(1+P^2W)P^{\varepsilon}. \tag{10.3}$$

In view of the minor arcs $\mathfrak{m}_q(\delta)$ defined in (5.1), we introduce the set $\mathscr{C}_\delta$ as

$$\mathscr{C}_\delta = \left\{(Y,W) : 1\le Y\le Q^{1/2-\delta}, \frac{Q^{-\delta}}{YQ} \ll W \ll \frac{Q^{\delta}}{YQ^{1/2}} \text{ or } Q^{1/2-\delta}\le Y\le Q, W \ll \frac{Q^{\delta}}{YQ^{1/2}}\right\}. \tag{10.4}$$

Every pair $(q,\vec{w})$ occurring in (10.2) satisfies $Y \le q \le 2Y, W \le |\vec{w}| \le 2W$ for some $(Y,W)\in\mathscr{C}_\delta$. Therefore we have

$$N_1(P,\delta) \ll P^{\varepsilon} \max_{(Y,W)\in\mathscr{C}_\delta} \left| \sum_{q\asymp Y} q^{-s} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\le V}} D_q(\mathbf{u}) \int_{|\vec{w}|\asymp W} p_{1,q}(\vec{w}) I_q(\vec{w},\mathbf{u})d\vec{w}\right| + P^{-N},$$

where, by a slight abuse of notation, we embrace both of the conditions $W \le |\vec{w}| \le 2W, \vec{w}\in\mathfrak{m}_q(\delta)$ in the notation $|\vec{w}| \asymp W$, so that the implicit constants are always in $[1,2]$, and are usually 1 and 2, but may differ when W is almost as large or as small as possible.

Writing $q = q_1q_2$ where q_1 is square free, q_2 is square-full with $q_1 \asymp Y_1, q_2 \asymp Y_2$ and $Y_1Y_2 \asymp Y$, we can apply Lemma 6.2 to the $\vec{w}$-integral and Lemma 7.5 to $D_q(\mathbf{u})$ to obtain the bound

$$\begin{aligned} N_1(P,\delta) \ll P^{\varepsilon} \max_{\substack{Y_1Y_2\asymp Y \\ (Y,W)\in\mathscr{C}_\delta}} & Y^{-s/2}QW^2P^s(1+P^2W)^{-s/2-1} \\ & \times \sum_{q_i\asymp Y_i} \sum_{|\mathbf{u}|\le V} \gcd(\mathscr{F}^*(\mathbf{u}),q_1)^{1/2}\gcd(\mathbf{u},q_1)^{1/2}\gcd(q_2,(D_F\mathscr{F}^*(\mathbf{u}))^{\infty}). \end{aligned}$$

To estimate the $\mathbf{u}$-sum, we consider the contribution from $\mathscr{F}^*(\mathbf{u}) \ne 0$ and $\mathscr{F}^*(\mathbf{u}) = 0$ separately.

10.0.1. *Case I:* $\mathscr{F}^*(\mathbf{u}) \neq 0$. The contributions from $\mathscr{F}^*(\mathbf{u}) \neq 0$ can be bounded by

$$\begin{aligned}
&P^\varepsilon \max_{\substack{Y_1Y_2 \asymp Y\\ (Y,W)\in\mathscr{C}_\delta}} Y^{-s/2}QW^2P^s(1+P^2W)^{-s/2-1}\\
&\times \sum_{\substack{|\mathbf{u}|\le V\\ \mathscr{F}^*(\mathbf{u})\neq 0}} \sum_{q_i\asymp Y_i} \gcd(\mathscr{F}^*(\mathbf{u}),q_1)^{1/2}\gcd(\mathbf{u},q_1)^{1/2}\gcd(q_2,(D_F\mathscr{F}^*(\mathbf{u}))^\infty)\\
&\ll P^\varepsilon \max_{\substack{Y_1Y_2 \asymp Y\\ (Y,W)\in\mathscr{C}_\delta}} Y^{-s/2}QW^2P^s(1+P^2W)^{-s/2-1}\sum_{\substack{|\mathbf{u}|\le V\\ \mathscr{F}^*(\mathbf{u})\neq 0}} Y_1Y_2\\
&\ll P^\varepsilon \max_{(Y,W)\in\mathscr{C}_\delta} Y^{1-s/2}QW^2P^s(1+P^2W)^{-s/2-1}V^s\\
&\ll P^\varepsilon \max_{(Y,W)\in\mathscr{C}_\delta} Y^{1+s/2}QW^2(1+P^2W)^{s/2-1}. \qquad (10.5)
\end{aligned}$$

The term on the right hand side of (10.5) is maximum when $W \asymp Y^{-1}Q^{-1/2+\delta}$ and $Y \asymp Q$, which is

$$\ll P^\varepsilon Y^{1+s/2}QW^2(Q/Y)^{s/2-1}Q^{(s/2-1)\delta} \ll P^\varepsilon Q^{s/2-1+(s/2+1)\delta}.$$

By choosing δ sufficiently small (depending on ε and s), this contribution is bounded by

$$Q^{2+s/2+\varepsilon}P^{-4} \ll P^{s-4-(s-8)/3+\varepsilon},$$

handing us (10.1).

10.0.2. *Case II:* $\mathscr{F}^*(\mathbf{u}) = 0$. To deal with the contribution from $\mathscr{F}^*(\mathbf{u}) = 0$ terms, we use Lemma 9.1 to obtain

$$\#\{|\mathbf{u}| \le V : \mathscr{F}^*(\mathbf{u}) = 0, \mathbf{u} \neq \mathbf{0}\} \ll V^{s-2+\varepsilon}.$$

The contribution from $\mathscr{F}^*(\mathbf{u}) = 0$ terms is therefore bounded by

$$\begin{aligned}
&\max_{\substack{Y_1Y_2 \asymp Y\\ (Y,W)\in\mathscr{C}_\delta}} Y^{-s/2}QW^2P^s(1+P^2W)^{-s/2-1}\\
&\times \sum_{\substack{|\mathbf{u}|\le V\\ \mathscr{F}^*(\mathbf{u})= 0}} \sum_{q_i\asymp Y_i} \gcd(\mathscr{F}^*(\mathbf{u}),q_1)^{1/2}\gcd(\mathbf{u},q_1)^{1/2}\gcd(q_2,(D_F\mathscr{F}^*(\mathbf{u}))^\infty)\\
&\ll \max_{\substack{Y_1Y_2 \asymp Y\\ (Y,W)\in\mathscr{C}_\delta}} Y^{-s/2+\varepsilon}QW^2P^s(1+P^2W)^{-s/2-1}\sum_{q_i\asymp Y_i}\left(Y_1Y_2 + \sum_{\substack{|\mathbf{u}|\le V\\ \mathscr{F}^*(\mathbf{u})= 0\\ \mathbf{u}\neq\mathbf{0}}} (\mathbf{u},q_1)^{1/2}Y_1^{1/2}Y_2\right)\\
&\ll \max_{\substack{Y_1Y_2 \asymp Y\\ (Y,W)\in\mathscr{C}_\delta}} Y^{-s/2+\varepsilon}Y_1^{1/2}Y_2QW^2P^s(1+P^2W)^{-s/2-1}(Y_1^{1/2}\sum_{q_i\asymp Y_i}1 + \sum_{\substack{|\mathbf{u}|\le V\\ \mathscr{F}^*(\mathbf{u})= 0\\ \mathbf{u}\neq\mathbf{0}}}\sum_{q_i\asymp Y_i}(\mathbf{u},q_1)^{1/2})\\
&\ll \max_{\substack{Y_1Y_2 \asymp Y\\ (Y,W)\in\mathscr{C}_\delta}} P^\varepsilon Y^{-s/2}Y_1^{1/2}Y_2QW^2P^s(1+P^2W)^{-s/2-1}(Y_1^{3/2}Y_2^{1/2} + V^{s-2}Y_1Y_2^{1/2})\\
&\ll \max_{(Y,W)\in\mathscr{C}_\delta} Y^{2-s/2}QW^2P^{s+\varepsilon}(1+P^2W)^{-s/2-1} + Y^{s/2-1/2}QW^2P^{2+\varepsilon}(1+P^2W)^{s/2-3}. \qquad (10.6)
\end{aligned}$$

Since $s \geq 5$, the first term in (10.6) is maximal when $Y \asymp 1, W \asymp Q^{-1-\delta}$ or $Y \asymp Q^{1/2-\delta}, W \asymp Q^{-3/2+\delta} \asymp P^{-2+\delta}$, which is bounded by

$$\ll Q^{-1}P^{s+\varepsilon}(P^2/Q)^{-s/2-1}Q^{(s/2+1)\delta} + Q^{2-s/4}P^{s-4+\varepsilon}Q^{(s/2+1)\delta} \ll P^{s-4-(s-8)/3+\varepsilon}Q^{(s/2+1)\delta}$$

The second term in (10.6) is maximal when $W \asymp Y^{-1}Q^{-1/2+\delta}$ and further when $Y \asymp Q$, as the resulting power of Y in this expression is positive. Therefore, we have, $Y \asymp Q, W \asymp Q^{-3/2+\delta} \asymp P^{-2+\delta}$, which gives a bound

$$\ll Q^{s/2+1/2+\varepsilon}P^{-4+2+\varepsilon}Q^{(s/2-1)\delta} \ll P^{s-4-(s-8)/3+\varepsilon}Q^{(s/2-1)\delta}.$$

By choosing δ sufficiently small, the contributions from $\mathscr{F}^*(\mathbf{u}) = 0$ also satisfy (10.1).

11. The Error Term $N_2(P, \delta)$

In this section, we will establish the bounds in Lemma 5.3 for the term $N_2(P,\delta)$ in (5.4):

$$N_2(P,\delta) \ll P^{s-4-(s-10)/3-1/6+\varepsilon} \text{ for } s \geq 10,$$
$$N_2(P,\delta) \ll P^{s-4-(s-9)/3-1/15+\varepsilon} \text{ under GLH for } s \geq 9$$

when δ is sufficiently small (depending on ε and s). From Lemmas 5.2 and 6.1, we see that

$$N_2(P,\delta) = \sum_{\substack{d,k\in\mathbb{N} \\ \vec{c}\in\mathbb{Z}^2 \text{primitive} \\ \vec{r}=dk|\vec{c}|\asymp Q^{1/2}}} \omega\Big(\frac{\vec{r}}{Q^{1/2}}\Big) \sum_{\substack{1\leq q\leq Q/k \\ d|q \\ \gcd(q/d,k)=1}} q^{-s} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\leq \frac{q}{P}(1+P^2|\vec{w}|)P^{\varepsilon}}} S_{q,d\vec{c}}(\mathbf{u}) \int_{\mathfrak{m}_q} p_{2,\vec{r},k,q}(\vec{w}) I_q(\vec{w},\mathbf{u})\, d\vec{w} + O_{\varepsilon,s,N}(P^{-N}).$$

After dividing $d, k, \vec{c}, q, \vec{w}$ into dyadic ranges, we obtain

$$N_2(P,\delta) \ll_N P^{-N} + P^{\varepsilon} \max_{\substack{KDC\asymp Q^{1/2} \\ D\ll Y\ll Q/K \\ (Y,W)\in\mathscr{C}_\delta}} \Bigg| \sum_{\substack{d,k\in\mathbb{N} \\ \vec{c}\in\mathbb{Z}^2\text{primitive} \\ d\asymp D, k\asymp K, |\vec{c}|\asymp C}} \sum_{\substack{q\in\mathbb{N} \\ d|q, q\asymp Y \\ \gcd(q/d,k)=1}} q^{-s} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\leq V}} S_{q,d\vec{c}}(\mathbf{u}) \int_{|\vec{w}|\asymp W} p_{2,\vec{r},k,q}(\vec{w}) I_q(\vec{w},\mathbf{u}) d\vec{w}\Bigg|, \tag{11.1}$$

where $\mathscr{C}_\delta$ is defined in (10.4) and V is defined as (10.3). Here we again include the condition $\vec{w} \in \mathfrak{m}_q(\delta)$ in the implicit constants in the notation $|\vec{w}| \asymp W$.

Remark 11.1. One could hope to take advantage of cancellations over the $\vec{c}$-sum in the general style of Health-Brown–Pierce [16] and Northey-Vishe [28] by using Cauchy's inequality to write

$$\sum_{\substack{\vec{c}\text{ primitive} \\ \vec{r}=dk\vec{c}}} \omega\Big(\frac{\vec{r}}{Q^{1/2}}\Big) \sum_{\mathbf{u}\in\mathbb{Z}^s} S_{q,d\vec{c}}(\mathbf{u}) \int_{\vec{w}\in\mathfrak{m}_q(\delta)} p_{2,\vec{r},k,q}(\vec{w}) I_q(\vec{w},\mathbf{u})\, d\vec{w}$$
$$\ll \Bigg(\Big(\frac{Q^\delta}{dqk}\Big)^2 \sum_{\substack{\vec{c}\text{ primitive} \\ \vec{r}=dk\vec{c} \\ |\vec{r}|\asymp Q^{1/2}}} \int_{\vec{w}\in\mathfrak{m}_q(\delta)} \Bigg| \sum_{\mathbf{u}\in\mathbb{Z}^s} S_{q,d\vec{c}}(\mathbf{u}) p_{2,\vec{r},k,q}(\vec{w}) I_q(\vec{w},\mathbf{u}) \Bigg|^2 d\vec{w} \Bigg)^{1/2}.$$

After bounding the integral, one would have sums of type $\sum_{\vec{c}} S_{q,d\vec{c}}(\mathbf{u})S_{q,d\vec{c}}(\mathbf{u}')$, in which some extra cancellations might be obtained. We do not explore this here since the saving does not seem to be enough for $s = 9$.

11.1. **Case:** $V < 1$. We first begin by considering the case when $V < 1$, where only the term $\mathbf{u} = \mathbf{0}$ contributes. From the definition in (10.3), we see that the condition $V < 1$ implies that the pairs (Y, W) belong to the set $\mathscr{C}_\delta^-$ defined as follows:

$$\mathscr{C}_\delta^- = \Big\{(Y, W) : Y \ll Q^{1/2-\delta}, Y^{-1}Q^{-1-\delta} \ll W \ll Y^{-1}P^{-1-\delta} \\ \text{or } Q^{1/2-\delta} \ll Y \ll P^{1-\delta}, W \ll Y^{-1}P^{-1-\delta}\Big\}.$$

Therefore, the contribution from $V < 1$ terms on the right side of (11.1) can be bounded by

$$\max_{\substack{KDC \asymp Q^{1/2} \\ D \ll Y \ll Q/K \\ (Y,W)\in\mathscr{C}_\delta^-}} Y^{-s} \sum_{k\asymp K, d\asymp D} \sum_{|\vec{c}|\asymp C} \sum_{q\asymp Y} |S_{q,d\vec{c}}(\mathbf{0})| \int_{|\vec{w}|\asymp W} |p_{2,\vec{r},k,q}(\vec{w}) I_q(\vec{w}, \mathbf{0})| \, d\vec{w}. \tag{11.2}$$

The integral $|I_q(\vec{w}, \mathbf{0})|$ can be estimated by applying Lemma 6.3 together with the trivial bound $(1 + P^2W\lambda_{\vec{c}})^{-1/2} \le 1$. The exponential sum $|S_{q,d\vec{c}}(\mathbf{0})|$ can be estimated using the first inequality in Lemma 7.11. The sum in (11.2) can therefore be bounded by

$$\begin{aligned}
&\ll WQ^{-3/2}P^s(1 + P^2W)^{-(s-1)/2} \sum_{\substack{q\asymp Y \\ d\asymp D, d|q \\ \vec{c}\asymp C, k\asymp K}} q^{-s}|S_{q,d\vec{c}}(\mathbf{0})| \\
&\ll WQ^{-3/2}P^s(1 + P^2W)^{-(s-1)/2} \sum_{\substack{d\asymp D \\ k\asymp K}} \sum_{\substack{q\asymp Y \\ d|q}} \sum_{\vec{c}\asymp C} q^{-s}dq^{s/2+1} \gcd\Big(\frac{q}{d}, \det M_{\vec{c}}\Big)^{1/2} \\
&\ll WQ^{-3/2}P^s(1 + P^2W)^{-(s-1)/2} \sum_{\substack{d\asymp D \\ k\asymp K}} \sum_{\substack{q\asymp Y \\ d|q}} q^{-s}q^{s/2+1}d \sum_{x|\frac{q}{d}} x^{1/2} \sum_{\substack{\vec{c}\asymp C \\ x|\det(M_{\vec{c}})}} 1 \\
&\ll Y^\varepsilon WQ^{-3/2}P^s(1 + P^2W)^{-(s-1)/2}Y^{-s/2+1}D \sum_{\substack{d\asymp D \\ k\asymp K}} \sum_{\substack{q\asymp Y \\ d|q}} \sum_{x|\frac{q}{d}} C(1 + \frac{C}{x^{1/2}})x^{1/2},
\end{aligned}$$

where we used (9.3) in the last inequality. It follows that (11.2) is bounded by

$$\begin{aligned}
&\ll P^\varepsilon \max_{\substack{KDC\asymp Q^{1/2} \\ D\ll Y\ll Q/K \\ (Y,W)\in\mathscr{C}_\delta^-}} WQ^{-3/2}P^s(1 + P^2W)^{-(s-1)/2}DKY^{-s/2+1}Y\left(Y^{1/2}D^{-1/2}C + C^2\right) \\
&\ll P^\varepsilon \max_{\substack{DC\ll Q^{1/2} \\ D\ll Y\ll Q \\ (Y,W)\in\mathscr{C}_\delta^-}} WQ^{-1}P^s(1 + P^2W)^{-(s-1)/2}Y^{-s/2+2}\left(Y^{1/2}D^{-1/2} + C\right). \qquad (11.3)
\end{aligned}$$

To estimate the maximal value of this expression for $(Y, W) \in \mathscr{C}_\delta^-$, we consider the case $Y \ll Q^{1/2-\delta}$ and $Y \gg Q^{1/2-\delta}$ separately. We also choose δ sufficiently small (depending on ε and s). If $Y \ll Q^{1/2-\delta}$, then $\mathscr{C}_\delta^-$ implies that $P^2W \gg Q^{-\delta}$. As a result, the maximal in (11.3) is achieved when $W \asymp Y^{-1}Q^{-1-\delta}$ is the smallest (as $s \geq 2$), which hands us the bound (when δ is sufficiently small)

$$\begin{aligned}&\ll Y^{-1}Q^{-1}Q^{-1}P^{s+\varepsilon}(1+Q^{1/2}/Y)^{-(s-1)/2}Y^{-s/2+2}(Y^{1/2}D^{-1/2}+C)\\ &\ll P^{s+\varepsilon}Y^{1/2}Q^{-s/4-7/4}(Y^{1/2}D^{-1/2}+C).\end{aligned}$$

Under the conditions $Y \ll Q^{1/2-\delta}$ and $DC \ll Q^{1/2}$, the maximal is achieved when $Y \asymp Q^{1/2-\delta}, C \asymp Q^{1/2}$. Therefore, this contribution to (11.3) can be estimated by

$$\ll P^{s+\varepsilon}Q^{-s/4-1} \ll P^{s-(s+4)/3+\varepsilon} \ll P^{s-4-(s-8)/3+\varepsilon}$$

when δ is sufficiently small. If $Y \gg Q^{1/2-\delta}$, then (11.3) is maximal when $W \asymp Q^{-3/2}$ and $Y \asymp Q^{1/2-\delta}$ or $W \asymp Y^{-1}P^{-1}$ and $Y \asymp P$, whose contributes to (11.3) can be bounded by

$$\begin{aligned}&Q^{-5/2}Q^{-s/4+1}P^{s+\varepsilon}(Q^{1/4}+Q^{1/2}) + P^{-1}P^{-1}Q^{-1}P^{s+\varepsilon}P^{-s/2+2}(P^{1/2}+Q^{1/2})\\ &\ll Q^{-s/4-1}P^{s+\varepsilon} + Q^{-1/2}P^{s/2+\varepsilon} \ll P^{s-4-(s-8)/3+\varepsilon},\end{aligned}$$

for $s \geq 9$ when δ is sufficiently small. We have thus established the bounds in Lemma 5.3.

11.2. **Case $V \geq 1$: initial manipulations.** We prove the following result for non-singular intersections of two quadrics. The situation for $V \geq 1$ requires more work. To handle this case, we recall $\mathscr{C}_\delta$ in (10.4), V in (10.3) and introduce the notation $\mathscr{Q}_\delta$ to denote the set $\mathscr{C}_\delta \setminus \mathscr{C}_\delta^-$:

$$\mathscr{Q}_\delta = \left\{(K,D,C,R,Y,W) : KDC \asymp Q^{1/2},\ D \ll R \ll Y \ll \frac{Q}{K}, (Y,W) \in \mathscr{C}_\delta, V \geq 1\right\}. \tag{11.4}$$

Lemma 11.2. *Let $\varepsilon > 0$ and $\mathscr{Q}_\delta$ be as in (11.4). Then for δ sufficiently small (depending on ε and s) we have*

$$\begin{aligned}N_2(P,\delta) \ll P^\varepsilon \max_{\mathscr{Q}_\delta} \sum_{k\asymp K}\sum_{d\asymp D}\sum_{|\vec{c}|\asymp C}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V}}\sum_{\substack{r\asymp R\\ \gcd(k,r/d)=1\\ d|r|\mathfrak{P}^\infty}} r^{-s}|S_{r,d\vec{c}}(\mathbf{u})\Sigma(r,Y/R,W,\mathfrak{P};k,d,\vec{c},\mathbf{u})|\\ + O(P^{s-4-(s-8)/3+\varepsilon}),\end{aligned} \tag{11.5}$$

where $\mathfrak{P} = \mathfrak{P}(d,\vec{c},\vec{\mathrm{F}})$ is any non-zero integer satisfying $d \mid \mathfrak{P}$, and

$$\Sigma(r,Y_1,W,\mathfrak{P};k,d,\vec{c},\mathbf{u}) = \int_{|\vec{\mathrm{w}}|\asymp W}\sum_{\substack{q_1\asymp Y_1\\ (q_1,k\mathfrak{P})=1}} q_1^{-s}S_{q_1,\vec{c}}(\mathbf{u})p_{2,dk\vec{c},k,q_1r}(\vec{\mathrm{w}})I_{q_1r}(\vec{\mathrm{w}},\mathbf{u})d\vec{\mathrm{w}}.$$

Proof. In Section 11.1, we showed that the contribution to (11.1) from $V < 1$ is $O(P^{s-4-(s-8)/3+\varepsilon})$ when δ is small enough, which is satisfactory. The remaining contribution, namely that from $V \geq 1$ is

$$\ll P^\varepsilon \max_{\mathscr{Q}_\delta}\sum_{k\asymp K}\sum_{d\asymp D}\sum_{|\vec{c}|\asymp C}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V}} N_2(Y,W;k,d,\vec{c},\mathbf{u}),$$

where

$$N_2(Y,W;k,d,\vec{c},\mathbf{u}) := \sum_{\substack{q\asymp Y\\ d|q\\ \gcd(q/d,k)=1}} q^{-s}S_{q,d\vec{c}}(\mathbf{u})\int_{|\vec{\mathrm{w}}|\asymp W} p_{2,dk\vec{c},k,q}(\vec{\mathrm{w}})I_q(\vec{\mathrm{w}},\mathbf{u})d\vec{\mathrm{w}}.$$

Here we again include the condition $\vec{\mathrm{w}} \in \mathfrak{m}_q(\delta)$ in the implicits constants in $|\vec{\mathrm{w}}| \asymp W$. We split $q = q_1 r$, where $\gcd(q_1, \mathfrak{P}k) = 1$ and $r \mid \mathfrak{P}^\infty$. Using $d \mid \mathfrak{P}$ and multiplicativity of the exponential sums, we can write

$S_{q,d\vec{c}}(\mathbf{u}) = S_{r,d\vec{c}}(\mathbf{u})S_{q_1,\vec{c}}(\mathbf{u})$ with $(q_1, \mathfrak{P}) = 1$ and $r \mid \mathfrak{P}^\infty$. Since $d \mid q$, we must have $d \mid r$ and the result follows by splitting r into dyadic ranges $r \asymp R$ with $D \ll R \ll Y/K$. □

We will choose $\mathfrak{P}$ differently depending on whether $\vec{c}$ is a good pair or a bad pair, in order to estimate $\Sigma(r, Y_1, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u})$.

Lemma 11.3 (Type I primes)**.** *Under the notation of Lemma 11.2, we have the following:*

(1) Let $\vec{c}$ is good and let $\mathfrak{P} = dD_F \det M_{\vec{c}}$. Then unconditionally we have the following bound

$$\Sigma(r, Y_1, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u}) \ll WQ^{-3/2}P^{s+\varepsilon}(1 + P^2W)^{-(s-1)/2}(1 + P^2W\lambda_{\vec{c}})^{-1/2} \times \begin{cases} Y_1^{-s/2+3/2} & 2 \nmid s, \\ Y_1^{-s/2+1} & 2 \mid s, F^*_{\vec{c}}(\mathbf{u}) \neq 0, \\ Y_1^{-s/2+2} & 2 \mid s, F^*_{\vec{c}}(\mathbf{u}) = 0. \end{cases} \tag{11.6}$$

If we further assume the generalized Lindelöf hypothesis for Dirichlet L-functions, then for $2 \nmid s$ and $F^*_{\vec{c}}(\mathbf{u}) \neq \square$, we may obtain

$$\Sigma(r, Y_1, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u}) \ll WQ^{-3/2}P^{s+\varepsilon}(1 + P^2W)^{-(s-1)/2}(1 + P^2W\lambda_{\vec{c}})^{-1/2}Y_1^{-s/2+1}. \tag{11.7}$$

(2) Suppose $\vec{c}$ is bad and let $\mathfrak{P} = dD_F$. Then we have

$$\Sigma(r, Y_1, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u}) \ll WQ^{-3/2}P^{s+\varepsilon}(1 + P^2W)^{-(s-1)/2}Y_1^{-s/2+3/2+\mathbb{1}_{2\mid s}/2},$$

if $Q^*_{\vec{c}}(\mathbf{u}') \neq 0$ or $((S^{-1})^t\mathbf{u})_s \neq 0$, and otherwise,

$$\Sigma(r, Y_1, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u}) \ll WQ^{-3/2}P^{s+\varepsilon}(1 + P^2W)^{-(s-1)/2}Y_1^{-s/2+5/2+\mathbb{1}_{2\mid s}/2}.$$

Proof. We write $\Sigma(r, Y_1) = \Sigma(r, Y_1, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u})$ for short.

(1) Let $\vec{c}$ be a good pair. Using Lemma 6.3 with $j = 0$, we have

$$\begin{aligned} |\Sigma(r, Y_1)| &\ll \sum_{\substack{q_1 \asymp Y_1 \\ (q_1, k\mathfrak{P})=1}} q_1^{-s}|S_{q_1,\vec{c}}(\mathbf{u})| \int_{|\vec{w}| \asymp W} |p_{2,dk\vec{c},q_1r}(\vec{w})I_{q_1r}(\vec{w}, \mathbf{u})| d\vec{w} \\ &\ll WQ^{-3/2}P^s(1 + P^2W)^{-(s-1)/2}(1 + P^2W\lambda_{\vec{c}})^{-1/2} \sum_{\substack{q_1 \asymp Y_1 \\ (q_1, k\mathfrak{P})=1}} q_1^{-s}|S_{q_1,\vec{c}}(\mathbf{u})|. \end{aligned} \tag{11.8}$$

Using Lemma 7.6, for $2 \mid s$ we have

$$\sum_{\substack{q_1 \asymp Y_1 \\ (q_1, k\mathfrak{P})=1}} q_1^{-s}|S_{q_1,\vec{c}}(\mathbf{u})| \ll \sum_{\substack{q_1 \asymp Y_1 \\ (q_1, k\mathfrak{P})=1}} q_1^{-s/2}(F^*_{\vec{c}}(\mathbf{u}), q_1) \ll Y_1^{-s/2+1+\mathbb{1}_{F^*_{\vec{c}}(\mathbf{u})=0}+\varepsilon}.$$

Similarly, for $2 \nmid s$, $F^*_{\vec{c}}(\mathbf{u}) \neq 0$ we have

$$\sum_{\substack{q_1 \asymp Y_1 \\ (q_1, k\mathfrak{P})=1}} q_1^{-s}|S_{q_1,\vec{c}}(\mathbf{u})| \ll \sum_{\substack{q_1 \asymp Y \\ (q_1, k\mathfrak{P})=1}} Y_1^{-s/2+1/2}(F^*_{\vec{c}}(\mathbf{u}), q_1)^{1/2} \ll Y_1^{-s/2+3/2+\varepsilon}.$$

If $2 \nmid s$ and $F^*_{\vec{c}}(\mathbf{u}) = 0$, we see by Lemma 7.6 that $S_{p^k,\vec{c}}(\mathbf{u}) = 0$ for $2 \nmid k$ and $S_{p^k,\vec{c}}(\mathbf{u}) \ll p^{ks/2+k}$ for $2 \mid k$. Therefore,

$$\sum_{\substack{q_1 \asymp Y_1 \\ (q_1,k\mathfrak{P})=1}} q_1^{-s}|S_{q_1,\vec{c}}(\mathbf{u})| \ll \sum_{\substack{q_1 \asymp Y_1 \\ p|q_1 \Rightarrow p^2|q_1 \\ (q_1,k\mathfrak{P})=1}} Y_1^{-s/2+1+\varepsilon} \ll P^\varepsilon Y_1^{-s/2+3/2+\varepsilon}.$$

This completes the proof of the unconditional bound (11.6) for $\Sigma(r, Y_1)$.

When s is odd and $F^*_{\vec{c}}(\mathbf{u}) \neq \square$, we can improve the estimate for $\Sigma(r, Y_1)$ by using cancellations in the sum $\sum_{q_1} S_{q_1,\vec{c}}(\mathbf{u})$, using Lemma 7.7. To be precise, we use integration by parts to write

$$\Sigma(r, Y_1) = \int_{|\vec{w}| \asymp W} \Big(\sum_{\substack{Y_1 \le q' \le q_1 \\ (q',k\mathfrak{P})=1}} (S_{q',\vec{c}}(\mathbf{u})) \Big) q_1^{-s} p_2(q_1 r, \vec{w}) I_{q_1 r}(\vec{w}, \mathbf{u}) \Big|_{Y_1}^{2Y_1} d\vec{w} \tag{11.9}$$

$$- \int_{|\vec{w}| \asymp W} \int_{Y_1 \le q_1 \le 2Y_1} \Big(\sum_{\substack{Y_1 \le q' \le q_1 \\ (q',k\mathfrak{P})=1}} (S_{q',\vec{c}}(\mathbf{u})) \Big) \partial_{q_1} \left(q_1^{-s} p_2(q_1 r, \vec{w}) I_{q_1 r}(\vec{w}, \mathbf{u}) \right) dq_1 d\vec{w}. \tag{11.10}$$

For any $q_1 \ll Y_1$, using Lemma 7.7, we have

$$\Big| \sum_{\substack{Y_1 \le q' \le q_1 \\ (q_1,k\mathfrak{P})=1}} S_{q',\vec{c}}(\mathbf{u}) \Big| \ll Y_1^{s/2+1+\varepsilon}. \tag{11.11}$$

The desired bound for the term on the right hand side of (11.9) follows upon applying Lemma 6.3 to estimate the integral over $\vec{w}$ and combining it with (11.11). For the term in (11.10), we change the order of integrals and use (11.11) to obtain

$$\int_{Y_1 \le q_1 \le 2Y_1} \Big(\sum_{\substack{Y_1 \le q' \le q_1 \\ (q',k\mathfrak{P})=1}} (S_{q',\vec{c}}(\mathbf{u})) \Big) \int_{|\vec{w}| \asymp W} \partial_{q_1} \left(q_1^{-s} p_2(q_1 r, \vec{w}) I_{q_1 r}(\vec{w}, \mathbf{u}) \right) d\vec{w}\, dq_1$$

$$\ll Y_1^{s/2+1+\varepsilon} \int_{Y_1 \le q_1 \le 2Y_1} \left| \int_{|\vec{w}| \asymp W} \partial_{q_1} \left(q_1^{-s} p_2(q_1 r, \vec{w}) I_{q_1 r}(\vec{w}, \mathbf{u}) \right) d\vec{w} \right| dq_1. \tag{11.12}$$

From the chain rule with respect to q for a fixed $\vec{w}$, we have

$$\begin{aligned} &\partial_{q_1} \left(q_1^{-s} p_2(q_1 r, \vec{w}) I_{q_1 r}(\vec{w}, \mathbf{u}) \right) \\ &= -q_1^{-s-1} p_2(q_1 r, \vec{w}) I_{q_1 r}(\vec{w}, \mathbf{u}) + q_1^{-s} I_{q_1 r}(\vec{w}, \mathbf{u}) r \partial_q p_2(q_1 r, \vec{w}) + q_1^{-s} p_2(q_1 r, \vec{w}) r \partial_q I_{q_1 r}(\vec{w}, \mathbf{u}). \end{aligned} \tag{11.13}$$

By substituting (11.13) into (11.12), we apply Lemma 6.3 to estimate the contribution from the first two terms of (11.13), and Lemma 6.4 to estimate the contribution from the final term in (11.13). This yields the desired bound for the expression in (11.10), under the hypothesis of Lemma 7.7, allowing us to conclude (11.7).

(2) When $\vec{c}$ is a bad pair, Lemma 7.9 hands us

$$\sum_{\substack{q_1 \asymp Y_1 \\ (q_1,dD_F)=1}} q_1^{-s} S_{q_1,\vec{c}}(\mathbf{u}) \ll \sum_{q_1 \asymp Y_1} q_1^{-s/2-1/2+1 \mathbb{1}_{2|s}/2} \gcd(q_1, Q^*_{\vec{c}}(\mathbf{u}')) \gcd(q_1, ((S^{-1})^t \mathbf{u})_s).$$

The result follows by summing over q_1 depending on whether $Q^*_{\vec{c}}(\mathbf{u}') = 0$ or $((S^{-1})^t \mathbf{u})_s = 0$, along with an application of (11.8) with $\lambda_{\vec{c}} = 0$. $\square$

Next, we need to consider the r-sum appearing in Lemma 11.2. An easy bound for the r-sum is given in following lemma.

Lemma 11.4. *Let $A > 0$ be some large fixed number. For any $d \asymp D, 1 \ll R \ll Q$, we have uniformly for $\mathfrak{P} \ll Q^A$*

$$\sum_{\substack{r \asymp R \\ d|r \\ r|\mathfrak{P}^\infty}} |S_{r,d\vec{c}}(\mathbf{u})| \ll Q^\varepsilon D^{1/2} R^{s/2+3/2}$$

Proof. The lemma follows from Lemma 7.11 and that the number of $r \asymp R$ with $r \mid \mathfrak{P}^\infty$ is $O(Q^\varepsilon)$. □

Lemma 11.4 turns out to be insufficient to compensate for our weaker bounds on exponential sums modulo powers of good primes of type II, as well as powers of bad primes. To address this, we combine 7.8 and Lemma 7.11 with Lemma 9.2 to obtain the following improved bound, which takes advantage of the average over $\mathbf{u}$.

Lemma 11.5. *Let $\varepsilon > 0$. Under the notation of Lemma 11.2, for δ sufficiently small (depending on ε and s) any $(K, D, C, R, Y, W) \in \mathscr{Q}_\delta$, we have*

$$\begin{aligned}
&Y^{-s/2+1}R^{-s/2-1}WQ^{-3/2}P^s(1+P^2W)^{-(s-1)/2} \\
&\times \sum_{\substack{d\asymp D \\ k \asymp K}} \sum_{\substack{\vec{c}\in\mathbb{Z}^2 \\ \text{good,primitive} \\ |\vec{c}|\asymp C}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\le V}} \sum_{\substack{r\asymp R \\ d|r \\ r|(dD_F \det M_{\vec{c}})^\infty}} |S_{r,d\vec{c}}(\mathbf{u})|(1+P^2W\lambda_{\vec{c}})^{-1/2} \\
&\ll P^{s-4-(s-9)/3-1/15+\varepsilon}.
\end{aligned}$$

Proof. For any fixed $k, d, \vec{c}, \mathbf{u}$, we write $r = r_2 r_3$ where $r_2 \mid \det(M_{\vec{c}})^\infty$, $(r_2, dD_F) = 1$ and r_2 is free of fifth powers, while r_3 consists of the remaining factors, i.e. r_3 consists of numbers whose prime factors divide dD_F and all 5-full numbers. Note that the condition $d \mid r$ implies $d \mid r_3$. We further split the r-sum into $O(R^\varepsilon)$ sums over r_2, r_3 with with $r_i \asymp R_i$ with $R_2R_3 \asymp R$, so that it is enough to bound

$$\begin{aligned}
\mathscr{S} =& Y^{-s/2+1}WQ^{-3/2}P^s(1+P^2W)^{-(s-1)/2} \\
&\times \sum_{\substack{d\asymp D \\ k\asymp K}} \sum_{\substack{\vec{c}\in\mathbb{Z}^2 \\ \text{primitive} \\ \vec{c}\text{ good} \\ |\vec{c}|\asymp C}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\le V}} \sum_{\substack{r_3\asymp R_3 \\ d|r_3 \\ r_3|(dD_F\det M_{\vec{c}})^\infty}} \sum_{\substack{r\asymp R \\ d|r \\ r_2|(\det M_{\vec{c}})^\infty \\ (r_2, r_3dD_F)=1 \\ r_2 \text{ fifth power free}}} (R_2R_3)^{-s/2-1}|S_{r,d\vec{c}}(\mathbf{u})|(1+P^2W\lambda_{\vec{c}})^{-1/2}.
\end{aligned}$$

By multiplicativity, we write $S_{r,d\vec{c}}(\mathbf{u}) = S_{r_2,\vec{c}}(\mathbf{u})S_{r_3,d\vec{c}}(\mathbf{u})$ and using 7.8 and Lemma 7.11 we obtain

$$|S_{r_2,\vec{c}}(\mathbf{u})| \ll r_2^{1+s/2}\gcd(r_2, ((S^{-1})^t\mathbf{u})_s, Q^*_{\vec{c}}(\mathbf{u}'))^{1/2} \tag{11.14}$$

$$|S_{r_3,d\vec{c}}(\mathbf{u})| \ll dr_3^{s/2+1}\gcd(r_3/d, ((S^{-1})^t\mathbf{u})_s, \det(M_{\vec{c}}))^{1/2}. \tag{11.15}$$

We arrange the order of summation as

$$\sum_k \sum_d \sum_{r_3} \sum_{\vec{c}} \sum_{r_2} \sum_{\mathbf{u}} (\cdots).$$

Combining our bounds in (11.14) and (11.15), we see that

$$\sum_{\substack{d|r_3\\ r_3\asymp R_3}}\sum_{|\vec{c}|\asymp C}\sum_{r_2\asymp R_2}\sum_{\mathbf{u}}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V}}(R_2R_3)^{-s/2-1}|S_{r_2r_3,d\vec{c}}(\mathbf{u})|(1+P^2W\lambda_{\vec{c}})^{-1/2}$$

$$\ll D\sum_{\substack{r_3\asymp R_3\\ d|r_3}}\sum_{|\vec{c}|\asymp C}(1+P^2W\lambda_{\vec{c}})^{-1/2}\sum_{r_2\asymp R_2}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V}}\gcd(r_2,((S^{-1})^t\mathbf{u})_s,Q^*_{\vec{c}}(\mathbf{u}'))^{1/2}\gcd(r_3/d,((S^{-1})^t\mathbf{u})_s,\det(M_{\vec{c}}))^{1/2}$$

$$\ll D\sum_{\substack{r_3\asymp R_3\\ d|r_3\\ x_1|\frac{r_3}{d}}}\sum_{\substack{|\vec{c}|\asymp C\\ x_1|\det(M_{\vec{c}})}}(1+P^2W\lambda_{\vec{c}})^{-1/2}\sum_{\substack{r_2\asymp R_2\\ x_2|r_2}}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ x_1x_2|((S^{-1})^t\mathbf{u})_s\\ x_2|Q^*_{\vec{c}}(\mathbf{u}')}}(x_1x_2)^{1/2}. \tag{11.16}$$

We apply (9.2) with $q=x_1x_2, d=x_2$ to the inner sum of (11.16), along with the observation that $\prod_{p|x_2}p\gg x_2^{1/4}$, since x_2 is free of fifth powers, to obtain

$$\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ x_1x_2|((S^{-1})^t\mathbf{u})_s\\ x_2|Q^*_{\vec{c}}(\mathbf{u}')}}1\ll V^{s-2}\min\Big(V+\frac{V^2}{x_1x_2},(1+\frac{V}{x_2^{1/4}})^2\Big)\ll V^{s-2}+\frac{V^s}{(x_1x_2)^{1/2}}+\min\Big(V^{s-1},\frac{V^s}{x_2^{1/2}}\Big).$$

We now consider the cases $R_2\le V^2$ and $R_2\ge V^2$ separately. If $R_2\le V^2$, we apply (9.3) with $N=P^2W$ to estimate the sum over $\vec{c}$, and obtain

$$\begin{aligned}
&\sum_{\substack{r_3\asymp R_3\\ x_1|\frac{r_3}{d}}}\sum_{\substack{|\vec{c}|\asymp C\\ x_1|\det(M_{\vec{c}})}}(1+P^2W\lambda_{\vec{c}})^{-1/2}\sum_{\substack{r_2\asymp R_2\\ x_2|r_2}}(x_1x_2)^{1/2}(V^s/(x_1x_2)^{1/2}+V^{s-1})\\
&\ll P^{\varepsilon}\sum_{\substack{r_3\asymp R_3\\ x_1|\frac{r_3}{d}}}C(1+C/(x_1(1+P^2W))^{1/2})(V^s+x_1^{1/2}R_2^{1/2}V^{s-1})\\
&\ll P^{\varepsilon}\sum_{\substack{r_3\asymp R_3\\ x_1|\frac{r_3}{d}}}C^2(V^s+R_2^{1/2}V^{s-1})(1+P^2W)^{-1/2}+\sum_{\substack{r_3\asymp R_3\\ x_1|\frac{r_3}{d}}}C(V^s+x_1^{1/2}R_2^{1/2}V^{s-1})\\
&\ll P^{\varepsilon}\Big((R_3/D)^{1/5}C^2V^s(1+P^2W)^{-1/2}+(R_3/D)^{1/5}C(V^s+(R_3R_2/D)^{1/2}V^{s-1})\Big).
\end{aligned}\tag{11.17}$$

If $R_2 \geq V^2$, then the contribution from divisors $x_2 \leq V^2$ can again be bound analogously as before. For the contribution from $x_2 \geq V^2$, we apply (9.3) with $N = P^2W$ to obtain

$$\begin{aligned}
&\sum_{\substack{r_3 \asymp R_3 \\ x_1 | \frac{r_3}{d}}} \sum_{\substack{|\vec{c}| \asymp C \\ x_1 | \det(M_{\vec{c}})}} (1 + P^2 W \lambda_{\vec{c}})^{-1/2} \sum_{\substack{r_2 \asymp R_2 \\ x_2 | r_2 \\ x_2 \gg V^2}} ((x_1 x_2)^{1/2} V^{s-2} + x_1^{1/2} V^s) \\
&\ll \sum_{\substack{r_3 \asymp R_3 \\ x_1 | \frac{r_3}{d}}} C(1 + C/(x_1(1+P^2W))^{1/2})((x_1 R_2)^{1/2} V^{s-2} + x_1^{1/2} V^s) \\
&\ll P^{\varepsilon} (R_3/D)^{1/5} \Big(C^2 (V^s + R_2^{1/2} V^{s-2})(1 + P^2W)^{-1/2} + C((R_3/D)^{1/2} V^s + (R_3 R_2/D)^{1/2} V^{s-2}) \Big) \\
&\ll P^{\varepsilon} (R_3/D)^{1/5} \Big(C^2 (V^s + R_2^{1/2} V^{s-2})(1 + P^2W)^{-1/2} + C (R_3 R_2/D)^{1/2} V^{s-1} \Big).
\end{aligned} \tag{11.18}$$

In the last inequality, we have used

$$(R_3/D)^{1/2} V^s \ll (R_3/D)^{1/2} R_2^{1/2} V^{s-1},$$

since $R_2 \geq V^2$. Combining (11.17) and (11.18) together, (11.16) is bounded by

$$\ll P^{\varepsilon} D (R_3/D)^{1/5} \Big(C^2 (V^s + (R_2 R_3/D)^{1/2} V^{s-2})(1 + P^2W)^{-1/2} + C(V^s + (R_3 R_2/D)^{1/2} V^{s-1}) \Big).$$

After summing over d, k, and using the fact that $DKC \asymp Q^{1/2}$, we obtain

$$\begin{aligned}
&DK(R_3/D)^{1/5} \Big(C^2 (V^s + (R_2 R_3/D)^{1/2} V^{s-2})(1 + P^2W)^{-1/2} + C(V^s + (R_3 R_2/D)^{1/2} V^{s-1}) \Big) \\
&\ll Q^{1/2} \Big(C(V^s (R_3/D)^{1/5} + C R_2^{1/2} (R_3/D)^{7/10} V^{s-2})(1 + P^2W)^{-1/2} + V^s (R_3/D)^{1/5} + R_2^{1/2} (R_3/D)^{7/10} V^{s-1} \Big) \\
&\ll Q^{1/2} \Big((Q^{1/2} R^{1/5} V^s + Q^{1/2} R^{7/10} V^{s-2})(1 + P^2W)^{-1/2} + R^{1/5} V^s + R^{7/10} V^{s-1} \Big).
\end{aligned}$$

By taking $V = Y P^{-1}(1 + P^2W) P^{\varepsilon}$, we see that the sum $\mathscr{S}$ we need to bound is

$$\begin{aligned}
\mathscr{S} &\ll Y^{-s/2+1} W Q^{-1} P^{s+\varepsilon} (1 + P^2W)^{-s/2} \\
&\quad \times \Big(Q^{1/2} R^{1/5} V^s + Q^{1/2} R^{7/10} V^{s-2} + R^{1/5} V^s (1 + P^2W)^{1/2} + R^{7/10} V^{s-1} (1 + P^2W)^{1/2} \Big) \\
&\ll Y^{s/2+1} W Q^{-1} (1 + P^2W)^{s/2} P^{\varepsilon} \\
&\quad \times \Big(Q^{1/2} R^{1/5} + Q^{1/2} R^{7/10} Y^{-2} P^2 (1 + P^2W)^{-2} + R^{1/5} (1 + P^2W)^{1/2} + R^{7/10} Y^{-1} P (1 + P^2W)^{-1/2} \Big) \\
&\ll Y^{s/2+1} W Q^{-1} (1 + P^2W)^{s/2} P^{\varepsilon} \\
&\quad \times \Big(Q^{1/2} R^{1/5} + Q^{1/2} R^{7/10} Y^{-2} P^2 (1 + P^2W)^{-2} + R^{7/10} Y^{-1} P (1 + P^2W)^{-1/2} \Big),
\end{aligned}$$

using the fact that $(1 + P^2W) \ll P^{\varepsilon}(Q/Y) \ll Q$ since $(Y, W) \in \mathscr{C}_\delta$ and δ is sufficiently small. As the exponents of W and R in the above expression are positive, the maximum over $\mathscr{Q}_\delta$ is attained when $W \asymp Y^{-1} Q^{-1/2} P^{\delta}$ which implies $P^2W \asymp (Q/Y) P^{\delta}$ and $R = Y$, in which case, the above becomes (when

δ is sufficiently small)

$$\begin{aligned}\ll & Y^{s/2}Q^{-3/2}(Q/Y)^{s/2}P^{\varepsilon}\left(Q^{7/10}+Q^{-3/2}Y^{7/10}P^{2}+Y^{7/10}Y^{-1}P(Q/Y)^{-1/2}\right)\\ \ll & P^{s+\varepsilon}Q^{-s/4-3/2}\left(Q^{7/10}+Q^{7/10-1/4}\right)\ll P^{s+\varepsilon}Q^{-s/4-4/5}\\ \ll & P^{s-(s/3+16/15)+\varepsilon}\ll P^{s-4-(s-9)/3-1/15+\varepsilon},\end{aligned} \tag{11.19}$$

by noting that the maximum of the expression in (11.19) is attained when $Y=Q$ as the resulting exponents of Y are positive and using the relation $Q=P^{4/3}$. □

11.3. **Contribution from good pairs $\vec{c}$: odd s.** Here, we will assume the generalized Lindeöf hypothesis as in Lemma 7.7. We further split the sum over $\mathbf{u}$ in (11.5) into two cases:

$$\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V}}=\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V\\F^*_{\vec{c}}(\mathbf{u})\neq\square}}+\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V\\F^*_{\vec{c}}(\mathbf{u})=\square}}. \tag{11.20}$$

We now define sums $N_{2,1}$ and $N_{2,2}$ corresponding contribution to (11.5) for good $\vec{c}$ from the two terms on the right hand side of (11.20).

11.3.1. *Case: $F^*_{\vec{c}}(\mathbf{u})\neq\square$.* When $F^*_{\vec{c}}(\mathbf{u})\neq\square$, we can use Lemma 11.3 to see that for δ sufficiently small and under GLH

$$\begin{aligned}N_{2,1}(P,\delta)&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta}\sum_{d\asymp D}\sum_{k\asymp K}\sum_{\substack{|\vec{c}|\asymp C\\\vec{c}\text{ good}}}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V\\F^*_{\vec{c}}(\mathbf{u})\neq\square}}\sum_{\substack{r\asymp R\\d|r\\r|(2dD_F\det M_{\vec{c}})^\infty}} r^{-s}|S_{r,d\vec{c}}(\mathbf{u})\Sigma(r,Y/R,W,\mathfrak{P};k,d,\vec{c},\mathbf{u})|\\&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta}Y^{-s/2+1}R^{-s/2-1}WQ^{-3/2}P^s(1+P^2W)^{-(s-1)/2}\\&\quad\times\sum_{d\asymp D}\sum_{k\asymp K}\sum_{|\vec{c}|\asymp C}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V\\F^*_{\vec{c}}(\mathbf{u})\neq\square}}\sum_{\substack{r\asymp R\\d|r\\r|(2dD_F\det M_{\vec{c}})^\infty}}|S_{r,d\vec{c}}(\mathbf{u})(1+P^2W\lambda_{\vec{c}})|^{-1/2},\end{aligned}$$

which is $\ll P^{s-4-(s-9)/3-1/15+\varepsilon}$ by using Lemma 11.5.

11.3.2. *Case: $F^*_{\vec{c}}(\mathbf{u})=\square$.* We now consider the sum over $\mathbf{u}$ for which $F^*_{\vec{c}}(\mathbf{u})=\square$. In this case, Lemma 11.3 gives

$$\begin{aligned}N_{2,2}(P,\delta)&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta}\sum_{d\asymp D}\sum_{k\asymp K}\sum_{\substack{|\vec{c}|\asymp C\\\vec{c}\text{ good}}}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V\\F^*_{\vec{c}}(\mathbf{u})=\square}}\sum_{\substack{r\asymp R\\d|r\\r|(2dD_F\det M_{\vec{c}})^\infty}} r^{-s}|S_{r,d\vec{c}}(\mathbf{u})\Sigma(r,Y/R,W,\mathfrak{P};k,d,\vec{c},\mathbf{u})|\\&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta}\sum_{d\asymp D}\sum_{k\asymp K}\sum_{\substack{|\vec{c}|\asymp C\\\vec{c}\text{ good}}}\sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\|\mathbf{u}|\le V\\F^*_{\vec{c}}(\mathbf{u})=\square}}\sum_{\substack{r\asymp R\\d|r\\r|(2dD_F\det M_{\vec{c}})^\infty}} R^{-s}|S_{r,d\vec{c}}(\mathbf{u})|\\&\quad\times WQ^{-3/2}P^s(1+P^2W)^{-(s-1)/2}(1+P^2W\lambda_{\vec{c}})^{-1/2}(Y/R)^{-s/2+3/2}.\end{aligned}$$

Using Lemma 11.4 for the r-sum we obtain

$$N_{2,2}(P,\delta) \ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} \sum_{d\asymp D} \sum_{k\asymp K} \sum_{\substack{|\vec{c}|\asymp C \\ \vec{c}\text{ good}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\le V \\ F^*_{\vec{c}}(\mathbf{u})=\square}} Y^{-s/2+3/2} W Q^{-3/2} P^s (1+P^2W)^{(1-s)/2} (1+P^2W\lambda_{\vec{c}})^{-1/2} D^{1/2}$$

$$\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} Y^{-s/2+3/2} D^{3/2} K W Q^{-3/2} P^s (1+P^2W)^{-(s-1)/2} \sum_{\substack{|\vec{c}|\asymp C \\ \vec{c}\text{ good}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s \\ |\mathbf{u}|\le V \\ F^*_{\vec{c}}(\mathbf{u})=\square}} (1+P^2W\lambda_{\vec{c}})^{-1/2}.$$

We now estimate the sum over primitive good $\vec{c}$ and $\mathbf{u}$ such that $F^*_{\vec{c}}(\mathbf{u}) = \square$ by considering whether $\mathscr{F}^*(\mathbf{u}) = 0$ or not. If $\mathscr{F}^*(\mathbf{u}) \neq 0$, we can bound the $\vec{c}, \mathbf{u}$ sum in two ways. First, we fix such a $\mathbf{u}$ and estimate the sum over $\vec{c}$ using [5, Theorem 5]. Here, for a fixed $\mathbf{u}$, we see $F^*_{\vec{c}}(\mathbf{u}) = z^2$ as a polynomial of degree $s-1$ in the $\vec{c}$ variable. Moreover, when $\mathscr{F}^*(\mathbf{u}) \neq 0$, $F^*_{\vec{c}}(\mathbf{u})$ is square-free and therefore this is an irreducible polynomial in z. Moreover, $F^*_{\vec{c}}(\mathbf{u}) = (z^{(s-1)/2})^2$ would be a homogeneous polynomial in $\vec{c}$ and z. Therefore, the hypothesis of [5, Theorem 5] is indeed satisfied and as a result we obtain:

$$\sum_{\substack{|\vec{c}|\asymp C \\ |\mathbf{u}|\le V \\ \mathscr{F}^*(\mathbf{u})\neq 0 \\ F^*_{\vec{c}}(\mathbf{u})=\square}} (1+P^2W\lambda_{\vec{c}})^{-1/2} \ll \sum_{\substack{|\mathbf{u}|\le V \\ \mathscr{F}^*(\mathbf{u})\neq 0}} \#\{|\vec{c}| \asymp C, \vec{c}\text{ primitive and good, } F^*_{\vec{c}}(\mathbf{u}) = \square\} \ll P^{\varepsilon} V^s C. \tag{11.21}$$

In a different approach, we first fix a good $\vec{c}$ and estimate the sum over $\mathbf{u}$ using a slight generalization of [15, Theorem 2] obtained in [35, Lemma 3.7]:

$$\#\{|\mathbf{u}| \le V : F^*_{\vec{c}}(\mathbf{u}) = z^2 : |\mathbf{u}| \le V\} \ll V^{s-1+\varepsilon}.$$

Combining it with the estimate for $\sum_{\vec{c}}(1+P^2W\lambda_{\vec{c}})^{-1/2}$ from (9.3) to obtain the second bound:

$$\sum_{\substack{|\vec{c}|\asymp C \\ |\mathbf{u}|\le V \\ \mathscr{F}^*(\mathbf{u})\neq 0 \\ F^*_{\vec{c}}(\mathbf{u})=\square}} (1+P^2W\lambda_{\vec{c}})^{-1/2} \ll P^{\varepsilon}\Big(V^{s-1}C + V^{s-1}C^2(1+P^2W)^{-1/2}\Big). \tag{11.22}$$

Combining (11.21) and (11.22) we see that

$$\sum_{\substack{|\vec{c}|\asymp C \\ |\mathbf{u}|\le V \\ \mathscr{F}^*(\mathbf{u})\neq 0 \\ F^*_{\vec{c}}(\mathbf{u})=\square}} (1+P^2W\lambda_{\vec{c}})^{-1/2} \ll P^{\varepsilon} V^{s-1} (1+P^2W)^{-1/2} C \min\{V(1+P^2W)^{1/2}, C\}. \tag{11.23}$$

If $\mathscr{F}^*(\mathbf{u}) = 0$, then we use Lemma 9.1 to estimate

$$\#\{|\mathbf{u}| \le V : \mathscr{F}^*(\mathbf{u}) = 0, F^*_{\vec{c}}(\mathbf{u}) = \square\} \ll \#\{|\mathbf{u}| \le V : \mathscr{F}^*(\mathbf{u}) = 0\} \ll V^{s-2+\varepsilon},$$

so that

$$\sum_{\substack{|\vec{c}|\asymp C \\ |\mathbf{u}|\le V \\ \mathscr{F}^*(\mathbf{u})=0 \\ F^*_{\vec{c}}(\mathbf{u})=\square}} (1+P^2W\lambda_{\vec{c}})^{-1/2} \ll P^{\varepsilon}\Big(V^{s-2}C + V^{s-2}C^2(1+P^2W)^{-1/2}\Big). \tag{11.24}$$

Combining (11.23) and (11.24), we obtain

$$\begin{aligned} N_{2,2}(P,\delta) &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+3/2}D^{3/2}KWQ^{-3/2}P^s(1+P^2W)^{-s/2}V^{s-1}C\min\{V(1+P^2W)^{1/2},C\} \\ &\quad+\max_{\mathscr{Q}_\delta} Y^{-s/2+3/2}D^{3/2}KWQ^{-3/2}P^s(1+P^2W)^{-s/2}V^{s-2}C((1+P^2W)^{1/2}+C) \\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+3/2}D^{1/2}WQ^{-1}P^s(1+P^2W)^{-s/2}C^{1/2}(1+P^2W)^{1/4}V^{s-1/2} \\ &\quad+\max_{\mathscr{Q}_\delta} Y^{-s/2+3/2}D^{1/2}WQ^{-1}P^s(1+P^2W)^{-s/2}((1+P^2W)^{1/2}+C)V^{s-2} \\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+3/2}D^{1/2}WQ^{-1}P^s(1+P^2W)^{-s/2} \\ &\qquad\times(C^{1/2}(1+P^2W)^{1/4}+((1+P^2W)^{1/2}+C)V^{-3/2})V^{s-1/2}. \end{aligned} \tag{11.25}$$

Comparing the powers of D, C and K in the above expression, since $KDC \asymp Q^{1/2}$, the maximum of the above expression is attained when $C \asymp Q^{1/2}, DK \asymp 1$ and this the above is

$$\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+3/2}WQ^{-1}P^s(1+P^2W)^{-s/2}(Q^{1/4}(1+P^2W)^{1/4}+((1+P^2W)^{1/2}+Q^{1/2})V^{-3/2})V^{s-1/2}.$$

Recall $V = YP^{-1}(1+P^2W)P^{\varepsilon}$ and $W \ll Y^{-1}Q^{-1/2}P^{\delta}$, we see that (11.25) again takes maximum when $W \asymp Y^{-1}Q^{-1/2}P^{\delta}$, which means $1+P^2W \asymp (Q/Y)P^{\delta}$ and $V \asymp P^{\varepsilon}P^{\delta}Q/P = P^{1/3+\varepsilon+\delta}$. Therefore, for δ sufficiently small (depending on ε and s) we have

$$\begin{aligned} N_{2,2}(P,\delta) &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+1/2}Q^{-3/2}P^s(1+P^2W)^{-s/2}(Q^{1/4}(1+P^2W)^{1/4}+Q^{1/2}V^{-3/2})V^{s-1/2} \\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+1/2}Q^{-3/2}P^s(Q/Y)^{-s/2}(Q^{1/4}(Q/Y)^{1/4}+Q^{1/2}P^{-1/2})P^{s/3-1/6} \\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+1/2}P^{s-2}(Q/Y)^{-s/2}(P^{1/3}(Q/Y)^{1/4}+P^{1/6})P^{s/3-1/6}. \end{aligned}$$

Since the power of Y appearing in the above is positive, this contribution is maximum when $Y \asymp Q$ and therefore for δ small enough

$$N_{2,2}(P,\delta) \ll Q^{-s/2+1/2}P^{s-2}(P^{1/3}+P^{1/6})P^{s/3-1/6+\varepsilon} \ll P^{s-4-(s-9)/3-1/6+\varepsilon}.$$

11.4. **Contribution from good pairs $\vec{c}$: even s.** Similar to before, we begin by splitting the sum over $\mathbf{u}$ into further two cases:

$$\sum_{|\mathbf{u}|\le V} = \sum_{\substack{|\mathbf{u}|\le V \\ F^*_{\vec{c}}(\mathbf{u})\neq 0}} + \sum_{\substack{|\mathbf{u}|\le V \\ F^*_{\vec{c}}(\mathbf{u})=0}}.$$

We again define sums $N_{2,1}$ and $N_{2,2}$, corresponding to the contribution to (11.5) for good $\vec{c}$, from the two terms on the right-hand side above.

11.4.1. *The case* $F^*_{\vec{c}}(\mathbf{u}) \neq 0$. Using Lemma 11.3, we see that *unconditionally* for even s and δ sufficiently small (depending on ε and s)

$$\begin{aligned}
N_{2,1}(P,\delta) &\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} \sum_{d\asymp D} \sum_{k\asymp K} \sum_{\substack{|\vec{c}|\asymp C\\ \vec{c}\text{ good}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V\\ F^*_{\vec{c}}(\mathbf{u})\neq 0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(2dD_F\det M_{\vec{c}})^\infty}} r^{-s} S_{r,d\vec{c}}(\mathbf{u}) \Sigma(r, Y/R, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u})\\
&\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} \sum_{d\asymp D} \sum_{k\asymp K} \sum_{|\vec{c}|\asymp C} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V\\ F^*_{\vec{c}}(\mathbf{u})\neq 0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(2dD_F\det M_{\vec{c}})^\infty}} R^{-s} |S_{r,d\vec{c}}(\mathbf{u})|\\
&\qquad \times WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}(1+P^2W\lambda_{\vec{c}})^{-1/2}(Y/R)^{-s/2+1},
\end{aligned}$$

which is $\ll P^{s-4-(s-9)/3-1/15+\varepsilon}$ again by using Lemma 11.5.

11.4.2. *The case* $F^*_{\vec{c}}(\mathbf{u}) = 0$. In this case, Lemma 11.3 gives

$$\begin{aligned}
N_{2,2}(P,\delta) &\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} \sum_{d\asymp D} \sum_{k\asymp K} \sum_{\substack{|\vec{c}|\asymp C\\ \vec{c}\text{ good}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V\\ F^*_{\vec{c}}(\mathbf{u})= 0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(2dD_F\det M_{\vec{c}})^\infty}} r^{-s} S_{r,d\vec{c}}(\mathbf{u}) \Sigma(r, Y/R, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u})\\
&\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} \sum_{d\asymp D} \sum_{k\asymp K} \sum_{|\vec{c}|\asymp C} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V\\ F^*_{\vec{c}}(\mathbf{u})= 0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(2dD_F\det M_{\vec{c}})^\infty}} R^{-s} |S_{r,d\vec{c}}(\mathbf{u})|\\
&\qquad \times WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}(1+P^2W\lambda_{\vec{c}})^{-1/2}(Y/R)^{-s/2+2}.
\end{aligned}$$

Using Lemma 11.4 for the r-sum we see that

$$N_{2,2}(P,\delta) \ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} \sum_{d\asymp D} \sum_{k\asymp K} \sum_{|\vec{c}|\asymp C} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V\\ F^*_{\vec{c}}(\mathbf{u})= 0}} Y^{-s/2+2} WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}(1+P^2W\lambda_{\vec{c}})^{-1/2} D^{1/2}R^{-1/2}$$

$$\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta} Y^{-s/2+2} R^{-1/2} D^{3/2} KWQ^{-3/2}P^s(1+P^2W)^{(1-s)/2} \sum_{|\vec{c}|\asymp C} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\leq V\\ F^*_{\vec{c}}(\mathbf{u})= 0}} (1+P^2W\lambda_{\vec{c}})^{-1/2}. \tag{11.26}$$

Now we focus on the $\vec{c}, \mathbf{u}$ sum. For a fixed good $\vec{c}$, $F^*_{\vec{c}}(\mathbf{u})$ is an irreducible quadratic form in $\mathbf{u}$ and therefore by [15, Theorem 2] we have

$$\#\{|\mathbf{u}| \leq V : F^*_{\vec{c}}(\mathbf{u}) = 0\} \ll V^{s-2+\varepsilon}. \tag{11.27}$$

However, this saving in the V variable together with (9.3) will not be enough. Therefore, we further split the sum over $\mathbf{u}$ into terms for which $\mathscr{F}^*(\mathbf{u}) \neq 0$ and $\mathscr{F}^*(\mathbf{u}) = 0$.

For fixed $\mathbf{u}$ such that $\mathscr{F}^*(\mathbf{u}) \neq 0$, we note that the polynomial $F^*_{\vec{c}}(\mathbf{u})$ must be a non-zero polynomial of degree $s-1$ in the $\vec{c}$ variable. $\mathscr{F}^*(\mathbf{u})$ is the discriminant of the polynomial $F^*_{\vec{c}}(\mathbf{u})$, seen as a polynomial in $\vec{c}$. Therefore, $F^*_{\vec{c}}(\mathbf{u})$ has at most $s-1$ primitive roots. As a result, we reach an alternate bound:

$$\#\{|\vec{c}| \asymp C, |\mathbf{u}| \leq V : \mathscr{F}^*(\mathbf{u}) \neq 0, F^*_{\vec{c}}(\mathbf{u}) = 0\} \ll V^{s+\varepsilon}. \tag{11.28}$$

Combining (11.27) with (9.3) and comparing this bound with (11.28), we have

$$\sum_{\substack{\vec{c}\in\mathbb{Z}^2, |\vec{c}|\asymp C\\ \text{good primitive}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s, |\mathbf{u}|\le V\\ F^*_{\vec{c}}(\mathbf{u})=0, \mathscr{F}^*(\mathbf{u})\neq 0}} (1+P^2W\lambda_{\vec{c}})^{-1/2} \ll V^{s-2+\varepsilon}C + V^{s-2+\varepsilon}\min\{C^2(1+P^2W)^{-1/2}, V^2\}$$

$$\ll V^{s-2+\varepsilon}C + V^{s-3/2+\varepsilon}C^{3/2}(1+P^2W)^{-3/8}$$

$$\ll P^{\varepsilon}C\Big(V^{s-2} + V^{s-3/2}C^{1/2}(1+P^2W)^{-3/8}\Big). \tag{11.29}$$

Now we consider the case $\mathscr{F}^*(\mathbf{u}) = 0$. If $\vec{c}$ is good and $s \ge 8$, then Lemma 5.4 shows that the variety $\mathscr{F}^*(\mathbf{u}) = F^*_{\vec{c}}(\mathbf{u}) = 0$ has projective dimension $s-3$ and no components of degree 1 or 3. Hence, applying Lemma 9.1 to each component of $\mathscr{F}^*(\mathbf{u}) = F^*_{\vec{c}}(\mathbf{u}) = 0$, we find

$$\{|\mathbf{u}| \le V : \mathscr{F}^*(\mathbf{u}) = F^*_{\vec{c}}(\mathbf{u}) = 0)\} \ll V^{s-3+\varepsilon}.$$

Therefore,

$$\sum_{\substack{\vec{c}\in\mathbb{Z}^2\\ |\vec{c}|\asymp C\\ \text{good primitive}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ F^*_{\vec{c}}(\mathbf{u})=0\\ \mathscr{F}^*(\mathbf{u})=0}} (1+P^2W\lambda_{\vec{c}})^{-1/2} \ll V^{s-3+\varepsilon}C(1+C(1+P^2W)^{-1/2}). \tag{11.30}$$

Combining (11.29) and (11.30) we see that

$$\sum_{\substack{\vec{c}\in\mathbb{Z}^2\\ |\vec{c}|\asymp C\\ \text{good primitive}}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ F^*_{\vec{c}}(\mathbf{u})=0}} (1+P^2W\lambda_{\vec{c}})^{-1/2}$$

$$\ll P^{\varepsilon}C\Big(V^{s-3/2}C^{1/2}(1+P^2W)^{-3/8} + V^{s-2} + V^{s-3}C(1+P^2W)^{-1/2}\Big).$$

Substituting these bounds in (11.26), we obtain

$$N_{2,2}(P,\delta) \ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+2}R^{-1/2}WD^{3/2}KCQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}$$
$$\times\Big(V^{s-3/2}C^{1/2}(1+P^2W)^{-3/8} + V^{s-2} + V^{s-3}C(1+P^2W)^{-1/2}\Big).$$

Recall that $DKC \asymp Q^{1/2}, V = YP^{-1}(1+P^2W)P^{\varepsilon}$ and $W \ll Y^{-1}Q^{-1/2}P^{\delta}$. We see that the maximum is reached when $W \asymp Y^{-1}Q^{-1/2}P^{\delta}, R \asymp 1$ and hence $1+P^2W \asymp P^{\delta}Q/Y, V \asymp P^{1/3+\varepsilon+\delta}$. Thus, for δ is sufficiently small (depending on ε and s), we have

$$N_{2,2}(P,\delta) \ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Y^{-s/2+1}D^{1/2}Q^{-3/2}P^s(Q/Y)^{(1-s)/2}C^{1/2}$$
$$\times\Big(P^{s/3-1/2}(Q/Y)^{-3/8} + P^{s/3-2/3} + P^{s/3-1}C^{1/2}(Q/Y)^{-1/2}\Big).$$

Further, the maximum is attained when $Y \asymp Q, R \asymp 1$ and upon further using $KDC \asymp Q^{1/2}$ and δ sufficiently small, we obtain

$$N_{2,2}(P,\delta) \ll P^{\varepsilon}\max_{\mathscr{Q}_\delta} Q^{-s/2+1+1/4}Q^{-3/2}P^s\Big(P^{s/3-1/2} + P^{s/3-2/3} + P^{s/3-1}Q^{1/4}\Big)$$
$$\ll P^{s+\varepsilon}Q^{-s/2-1/4}\Big(P^{s/3-1/2} + P^{s/3-2/3}\Big) \ll P^{s-s/3-2/3-1/6+\varepsilon}$$
$$\ll P^{s-4-(s-10)/3-1/6+\varepsilon}.$$

11.5. **Contribution from bad pairs $\vec{c}$.** In this case, we know $|\vec{c}| \ll 1$. We henceforth fix a bad pair $\vec{c}$ and estimate the minor arc contribution from this pair. We further split the sum over $\mathbf{u}$ in (11.5) into two cases:

$$\sum_{|\mathbf{u}|\le V} = \sum_{\substack{|\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')\neq 0\\ \text{or } ((S^{-1})^t\mathbf{u})_s\neq 0}} + \sum_{\substack{|\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')=((S^{-1})^t\mathbf{u})_s=0}} .$$

We again define $E_{2,1}$ and $E_{2,2}$, corresponding contribution to (11.5) for bad $\vec{c}$, from the two terms on the right hand side above. Note that when $\vec{c}$ is bad, we have $\lambda_{\vec{c}} = 0$.

11.5.1. *Case: $Q^*_{\vec{c}}(\mathbf{u}') \neq 0$ or $((S^{-1})^t\mathbf{u})_s \neq 0$.* Using Part (2) of Lemma 11.3 which requires $Q^*_{\vec{c}}(\mathbf{u}') \neq 0$ or $((S^{-1})^t\mathbf{u})_s \neq 0$, we see that

$$\begin{aligned} E_{2,1}(P,\delta) &\ll P^\varepsilon \max_{\mathscr{Q}_\delta, C\ll 1} \sum_{d\asymp D}\sum_{k\asymp K} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')\neq 0\\ \text{or } ((S^{-1})^t\mathbf{u})_s\neq 0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(dD_F)^\infty}} r^{-s} S_{r,d\vec{c}}(\mathbf{u})\Sigma(r, Y/R, W, \mathfrak{P}; k, d, \vec{c}, \mathbf{u}) \\ &\ll P^\varepsilon \max_{\mathscr{Q}_\delta, C\ll 1} \sum_{d\asymp D}\sum_{k\asymp K} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')\neq 0\\ \text{or } ((S^{-1})^t\mathbf{u})_s\neq 0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(dD_F)^\infty}} R^{-s}|S_{r,d\vec{c}}(\mathbf{u})| \\ &\qquad \times WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}(Y/R)^{-s/2+3/2+\mathbb{1}_{2|s}/2}. \end{aligned} \tag{11.31}$$

Our weak bound in Lemma 11.4 is insufficient due to the loss of the $d^{1/2}$ factor, and we need to use the improved bound from Lemma 7.12. Therefore, we need to further split the d and r-sum into $d = d_1d_2d_3$, where $\gcd(d_1d_2, D_F) = 1$, d_i pairwise co-prime, d_1d_2 is square-free with $\gcd(d_1, r/d_1) = 1$, $d_2^2 \mid r$ and d_3 consists of numbers whose prime divisors p satisfies $p \mid D_F$ or $p^2 \mid d_3$. Writing $r = d_1r_3$, with $d_2^2d_3 \mid r_3$ and $r_3 \mid (d_2d_3D_F)^\infty$, we see that

$$S_{r,d\vec{c}}(\mathbf{u}) = S_{d_1r_3,d_1d_2d_3\vec{c}}(\mathbf{u}) = S_{d_1,d_1\vec{c}}(\mathbf{u})S_{r_3,d_2d_3\vec{c}}(\mathbf{u}).$$

We will re-order the sums as

$$\sum_{d_2}\sum_{d_3}\sum_{r_3}\sum_{\mathbf{u}}\sum_{d_1}(\cdots),$$

and split sums $d_i \asymp D_i, r_3 \asymp R_3$ into dyadic ranges. From Lemma 7.12, we see that for $\mathbf{u} \neq \mathbf{0}$

$$\sum_{d_1\asymp D_1} |S_{d_1,d_1\vec{c}}(\mathbf{u})| \ll D_1^{s/2+3/2} \sum_{d_1\asymp D_1} \gcd(d_1,\mathbf{u})^{1/2} \ll D_1^{s/2+5/2+\varepsilon}, \tag{11.32}$$

and from Lemma 7.11, we have

$$|S_{r_3,d_2d_3\vec{c}}(\mathbf{u})| \ll D_2D_3R_3^{s/2+1}\gcd(r_3/d_2d_3, ((S^{-1})^t\mathbf{u})_s))^{1/2}. \tag{11.33}$$

Moreover, there are $O(R_3^\varepsilon)$ different values of r_3 for fixed d_2, d_3 and there are $O(D_2)$ numbers of $d_2 \asymp D_2$ and $O(D_3^{1/2+\varepsilon})$ numbers of $d_3 \asymp D_3$ for any $\varepsilon > 0$. From (11.32) and (11.33) we have

$$\sum_{d_2}\sum_{d_3}\sum_{r_3}\sum_{\mathbf{u}}\sum_{d_1} |S_{d_1r_1r_3,d_1d_2d_3\vec{c}}(\mathbf{u})| \ll P^\varepsilon D_1^{s/2+5/2}D_2D_3R_3^{s/2+1} \sum_{d_2,d_3}\sum_{r_3}\sum_{x_1|\frac{r_3}{d_2d_3}} \sum_{\substack{0\neq|\mathbf{u}|\le V\\ x_1|((S^{-1})^t\mathbf{u})_s}} x_1^{1/2}.$$

After an application of (9.2) to estimate the $\mathbf{u}$ sum, this is

$$\begin{aligned}&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} D_1^{s/2+5/2}D_2D_3R_3^{s/2+1}\sum_{d_2,d_3}\sum_{r_3}\sum_{x_1\mid r_3/d_2d_3}(x_1^{1/2}V^{s-1}+V^s)\\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} D_1^{s/2+5/2}D_2^2D_3^{3/2}R_3^{s/2+1}V^{s-1}(R_3^{1/2}D_2^{-1/2}D_3^{-1/2}+V)\\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} R^{s/2+3/2}D_1D_2^2D_3^{3/2}V^{s-1}(D_2^{-1/2}D_3^{-1/2}+VR_3^{-1/2}).\end{aligned}$$

Substituting back in (11.31), this contribution is

$$\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+3/2+\mathbb{1}_{2|s}/2}R^{-\mathbb{1}_{2|s}/2}WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}D_1D_2^2D_3^{3/2}KV^{s-1}(D_2^{-1/2}D_3^{-1/2}+VR_3^{-1/2})$$

$$\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+3/2+\mathbb{1}_{2|s}/2}R^{-\mathbb{1}_{2|s}/2}WQ^{-1}P^s(1+P^2W)^{(1-s)/2}D_2D_3^{1/2}V^{s-1}(D_2^{-1/2}D_3^{-1/2}+VR_3^{-1/2}),$$

since $D_1D_2D_3K\asymp Q^{1/2}$. Note that since $d_2^2d_3\mid r_3$, this contribution is

$$\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+3/2+\mathbb{1}_{2|s}/2}R^{-\mathbb{1}_{2|s}/2}WQ^{-1}P^s(1+P^2W)^{(1-s)/2}V^{s-1}(D_2^{1/2}+V).$$

When $2\nmid s$, contribution is maximum when $K=D_1=D_3=1$ and $D_2\asymp Q^{1/2}$ and is bounded by

$$\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+3/2}WQ^{-1}P^s(1+P^2W)^{(1-s)/2}V^{s-1}(Q^{1/4}+V).$$

As before, maximum value is reached when $W=Q^{-1/2+\delta}Y^{-1}$, and therefore $V=Q^{1/4+\varepsilon+\delta}$. Therefore, when δ is sufficiently small (depending on ε and s) this contribution is

$$\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+1/2}Q^{-3/2}P^s(Q/Y)^{(1-s)/2}Q^{s/4}\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1}P^sQ^{-s/4-1}\ll P^{s-4-(s-8)/3+\varepsilon},$$

which suffices for $s\geq 9$. When $2\mid s$, our contribution is

$$\begin{aligned}&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+2}R^{-1/2}WQ^{-1}P^s(1+P^2W)^{(1-s)/2}V^{s-1}(D_2^{1/2}+V)\\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+2}R^{-1/2}WQ^{-1+1/4}P^s(1+P^2W)^{(1-s)/2}V^{s-1},\end{aligned}$$

upon bounding $D_2^{1/2}+V\ll Q^{1/4+\varepsilon+\delta}$ and using that δ is sufficiently small. We next note that $d\mid r_3$ and $Y\leq Q/K$ therefore $Y/R\leq Q/(KD)\leq Q^{1/2}$. Upon further substituting $V=YP^{-1+\varepsilon}(1+P^2W)$, the expression is maximum when $W=Y^{-1}Q^{-1/2+\delta}$ and therefore $V\asymp Q^{1/4+\varepsilon+\delta}$. By choosing δ sufficiently small interms of ε and s, we see that the above is

$$\begin{aligned}&\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+3/2}WQ^{-1/2}P^s(1+P^2W)^{(1-s)/2}V^{s-1},\\ &\ll P^{\varepsilon}\max_{\mathscr{Q}_\delta,C\ll 1} Y^{-s/2+1/2}Q^{-1}P^s(Q/Y)^{(1-s)/2}Q^{s/4-1/4}\ll P^{s+\varepsilon}Q^{-s/4-3/4}\ll P^{s-4-(s-9)/3+\varepsilon},\end{aligned}$$

which suffices for $s\geq 10$.

11.5.2. *Case:* $Q^*_{\vec{c}}(\mathbf{u}') = ((S^{-1})^t\mathbf{u})_s = 0$. Using Lemma 11.3, we see that

$$\begin{aligned} E_{2,2}(P,\delta) &\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta, C\ll 1} \sum_{d\asymp D}\sum_{k\asymp K} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')=((S^{-1})^t\mathbf{u})_s=0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(dD_F)^\infty}} r^{-s} S_{r,d\vec{c}}(\mathbf{u})\Sigma(r,Y/R,W,\mathfrak{P};k,d,\vec{c},\mathbf{u}) \\ &\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta, C\ll 1} \sum_{d\asymp D}\sum_{k\asymp K} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')=((S^{-1})^t\mathbf{u})_s=0}} \sum_{\substack{r\asymp R\\ d|r\\ r|(dD_F)^\infty}} R^{-s}|S_{r,d\vec{c}}(\mathbf{u})| \\ &\qquad \times WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}(Y/R)^{-s/2+5/2+\mathbb{1}_{2|s}/2}. \end{aligned}$$

Using Lemma 11.4 for the r-sum, we see that

$$E_{2,2}(P,\delta) \ll P^{\varepsilon} \max_{\mathscr{Q}_\delta, C\ll 1} Y^{-s/2+5/2+\mathbb{1}_{2|s}/2} WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}D^{3/2}KR^{-1-\mathbb{1}_{2|s}/2} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^s\\ |\mathbf{u}|\le V\\ Q^*_{\vec{c}}(\mathbf{u}')=((S^{-1})^t\mathbf{u})_s=0}} 1.$$

To estimate the sum over $\mathbf{u}$, we note that the conditions $Q^*_{\vec{c}}(\mathbf{u}') = ((S^{-1})^t\mathbf{u})_s = 0$ define a variety of projective dimension $s-3$. As a result, using Lemma 9.1,

$$\#\{|\mathbf{u}|\le V : Q^*_{\vec{c}}(\mathbf{u}') = ((S^{-1})^t\mathbf{u})_s = 0\} \ll V^{s-3+\varepsilon}.$$

Thus

$$E_{2,2}(P,\delta) \ll P^{\varepsilon} \max_{\mathscr{Q}_\delta, C\ll 1} Y^{-s/2+5/2+\mathbb{1}_{2|s}/2} WQ^{-3/2}P^s(1+P^2W)^{(1-s)/2}D^{3/2}KR^{-1-\mathbb{1}_{2|s}/2}V^{s-3}.$$

With $V = \frac{Y}{P}(1+P^2W)P^{\varepsilon}$, the maximal is achieved at $W = Y^{-1}Q^{-1/2+\delta}$, which finally gives when δ is sufficiently small

$$\begin{aligned} E_{2,2}(P,\delta) &\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta, C\ll 1} Y^{-s/2+3/2+\mathbb{1}_{2|s}/2} Q^{-3/2}P^s(Q/Y)^{(1-s)/2}D^{1/2}R^{-1-\mathbb{1}_{2|s}/2}Q^{s/4-3/4} \\ &\ll P^{\varepsilon} \max_{\mathscr{Q}_\delta, C\ll 1} Q^{-s/4-3/4+\mathbb{1}_{2|s}/2}D^{1/2}K^{-1-\mathbb{1}_{2|s}/2}R^{-1-\mathbb{1}_{2|s}/2}P^s \\ &\ll Q^{-s/4-1+\mathbb{1}_{2|s}/4}P^{s+\varepsilon} \ll P^{s-4-(s-8-\mathbb{1}_{2|s})/3+\varepsilon}, \end{aligned}$$

using $Y \ll Q/K$, $D \ll Q^{1/2}$ and $RK \gg DK \asymp Q^{1/2}$. This is admissible as soon as $s \ge 9$ when δ is sufficiently small in terms of ε and s.

12. Heuristic comparison of δ-methods

In this section, we give a heuristic to allow comparison of our two-dimensional delta symbol to other existing methods. Based on the one-dimensional delta symbol and the two-dimensional Farey dissection over function fields, we consider an R-dimensional delta symbol method over $\mathbb{Q}$ to be any identity of the type

$$\delta_{\vec{n}} = \sum_{1\le q\le Q} \sideset{}{^*}\sum_{\vec{a} \bmod q} \int_{|\vec{w}|\le \frac{Q^{\epsilon}}{qQ^{\eta}}} p_{q,\vec{a}}(\vec{w}) e((\vec{a}/q+\vec{w})\cdot\vec{n})\, d\vec{w} + O_{\eta,N}(Q^{-N}), \tag{12.1}$$

for some $0<\eta\le 1/R$, some explicit smooth functions $p_{q,\vec{a}}$, and for all $\vec{n}\in\mathbb{Z}^R$ and all $Q,N,\epsilon>0$. We believe that many practitioners of the circle method have some intuition along the following lines.

Heuristic 1. If we apply (12.1) to a sequence of vectors $\vec{\mathrm{n}}$ which typically have size around M, then we should choose Q so that $M = Q^{1+\eta}$ holds. In this way, for $q \asymp Q$, the function $e((\vec{\mathrm{a}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{n}})$ does not oscillate very much as $\vec{\mathrm{w}}$ varies over the domain of integration. It is desirable for the efficacy of this approach that Q should be taken as small as possible.

According to the heuristic above, it may be desired for $Q = M^{1/(1+\eta)}$ to be as small as possible, in other words η should be as large as possible. However we have the restriction $\eta \leq 1/R$ in (12.1). The reason is that the numbers $\vec{\mathrm{a}}/q+\vec{\mathrm{w}}$ will have to run over essentially the entire unit box $[0,1]^R$ in order to approximate the function $\delta_{\vec{\mathrm{n}}}$ accurately in ℓ^∞ norm. By Khinchine's theorem, this requires $|\vec{\mathrm{w}}| \gg_\epsilon q^{-1}Q^{-1/R-\epsilon}$ for any $\epsilon > 0$, and hence $\eta \leq 1/R$ must hold. In particular, our form of the two-dimensional delta symbol in Theorem 1.2 with $\eta = 1/2$ has the optimal choice of $Q = M^{2/3}$.

To perform a Kloosterman refinement by taking advantage of cancellations in the sum over $\vec{\mathrm{a}}$, one needs to arrange that the function $p_{q,\vec{\mathrm{a}}}$ is the same for many different $\vec{\mathrm{a}}$. We partition $(\mathbb{Z}^R/q\mathbb{Z}^R)^*$ into equivalence classes $[\vec{\mathrm{a}}]$ such that $[\vec{\mathrm{a}}] = [\vec{\mathrm{b}}]$ exactly when $p_{q,\vec{\mathrm{a}}} = p_{q,\vec{\mathrm{b}}}$ as functions. We can hope to make use of cancellations in the sums $\sum_{\vec{\mathrm{b}}\in[\vec{\mathrm{a}}]} e((\vec{\mathrm{b}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{n}})$ to get additional savings. There may be a trivial class $\{\vec{\mathrm{a}} : p_{q,\vec{\mathrm{a}}}(\vec{\mathrm{w}}) = 0 \ \forall \vec{\mathrm{w}}\}$ on which $p_{q,\vec{\mathrm{a}}}$ vanishes; we shall exclude this class from our analysis as it contributes nothing to (12.1). We let A be the average size of a nontrivial class $[\vec{\mathrm{a}}]$, that is

$$A = \Big(\sum_{1\leq q\leq Q} \#\mathscr{A}_q\Big)^{-1}\Big(\sum_{1\leq q\leq Q}\sum_{C\in\mathscr{A}_q} \#C\Big), \qquad \mathscr{A}_q = \{C : C = [\vec{\mathrm{a}}] \text{ for some } \vec{\mathrm{a}} \text{ with } (\vec{\mathrm{a}}, q) = 1, p_{q,\vec{\mathrm{a}}} \neq 0\}.$$

Our Theorem 1.2 allows averages over $\vec{\mathrm{a}}$ with $A \asymp Q$. We mention that the methods in [28, 16, 31] effectively take the optimal values $\eta = 1/R$, $A = \#((\mathbb{Z}^R/q\mathbb{Z}^R)^*)$, but they can only be applied to situations where the exponential sum is an absolute square, due to the use of a classical major-arc/minor-arc decomposition rather than the δ-method. With this notation we give a heuristic for δ-methods in Diophantine problems.

Heuristic 2. Suppose we use (12.1) as a form of the circle method to count solutions to $\vec{\mathrm{F}} = \vec{0}$, where $\vec{\mathrm{F}}(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_R(\mathbf{x}))$ is a system of R polynomials in $\mathbf{x} \in \mathbb{Z}^s$ with $|\vec{\mathrm{F}}(\mathbf{x})| \ll M$. Then we should take $Q = M^{1/(1+\eta)}$ and the efficacy of (12.1) is roughly captured by the quantity

$$Q^{s/2}A^{-1/2},$$

where A is as defined above. A smaller value of this quantity suggests a more effective δ-method. In principle, a double Kloosterman refinement utilizing also the average over q might save a another factor of $Q^{1/2}$, although in practice, this saving may be less due to lack of good estimates for short character sums without GLH. Additionally one may want to apply differencing methods to exponential sums over $\vec{\mathrm{F}}$, such as van der Corput differencing. In such cases one can instead take M to be an upper bound for a system $\Delta_{\mathbf{h}}\vec{\mathrm{F}}(\mathbf{x})$ of differenced polynomials.

The reasoning behind Heuristic 2 is as follows. Using the delta symbol in (12.1) and summing over $\mathbf{x}$, we see that the number of $\mathbf{x}$ with $|\mathbf{x}| \leq P$, $\vec{\mathrm{F}} = \vec{0}$, is essentially given by

$$\sum_{q\leq Q}\sideset{}{^*}\sum_{\vec{\mathrm{a}} \bmod q} \int_{|\vec{\mathrm{w}}|\ll\frac{1}{qQ^\eta}} p_{q,\vec{\mathrm{a}}}(\vec{\mathrm{w}}) \sum_{|\vec{\mathrm{x}}|\leq P} e((\vec{\mathrm{a}}/q + \vec{\mathrm{w}}) \cdot \vec{\mathrm{F}}(\mathbf{x}))d\vec{\mathrm{w}}.$$

After any differencing steps one applies Poisson summation in the $\mathbf{x}$ variable to modulus q; the innermost sum becomes $\sum_{\mathbf{u}} q^{-s}S_q(\vec{\mathrm{a}}, \mathbf{u})I_q(\vec{\mathrm{w}}, \mathbf{u})$ for some exponential sums $S_q(\vec{\mathrm{a}}, \mathbf{u})$ and exponential integrals $I_q(\vec{\mathrm{w}}, \mathbf{u})$. The integral $I_q(\vec{\mathrm{w}}, \mathbf{u})$ allows a truncation of the $\mathbf{u}$ variables up to essentially $\frac{q}{P}(1+M|\vec{\mathrm{w}}|)$ (see e.g. Lemma 6.1 in our setting). The zero frequency $\mathbf{u} = \mathbf{0}$ will generally give the main term. For the non-zero frequencies, one may expect $I_q \ll P^s(1+M|\vec{w}|)^{-s/2}$ from stationary phase analysis and the bound $S_q \ll q^{s/2}$, assuming square-root cancellation of exponential sums. If we can further make use of the average over $\vec{a}$ to get a saving of $(\#[\vec{a}])^{-1/2}$, we may find that the non-zero frequencies contribute to a term of size roughly

$$q^{s/2}(1+M|\vec{w}|)^{s/2}(\#[\vec{a}])^{-1/2} \le (Q+M/Q^{\eta})^{s/2}(\#[\vec{a}])^{-1/2}.$$

This error term is of size $\gg Q^{s/2}(\#[\vec{a}])^{-1/2}$ as soon as $Q \ge M^{1/(1+\eta)} \gg M^{R/(R+1)}$. This predicts an error in our original problem of size

$$\sum_{q\le Q} \sum_{\vec{a} \bmod q}^{*} \int_{|\vec{w}|\ll \frac{1}{qQ^{\eta}}} p_{q,\vec{a}}(\vec{w}) O(Q^{s/2}(\#[\vec{a}])^{-1/2})\, d\vec{w}.$$

We generally expect $\sum_q \sum_{\vec{a}}^{*} \int |p_{q,\vec{a}}(\vec{w})|\, d\vec{w} \ll Q^{\epsilon}$ to hold and that we can replace $\#[\vec{a}]$ by its average A. Hence the quantity above should be around $Q^{s/2}A^{-1/2}$ as we posit above. Furthermore, if one caries out a double Kloosterman refinement, one may save another $Q^{1/2}$ from the q-sum.

Remark 12.1. It is tempting to interpret Heuristic 2 as a genuine prediction for the size of the error term in the best possible case. But we should remember that the differencing procedures referred to at the end of the heuristic might alter the actual error term. Moreover better than square-root cancellation in exponential sums may be available such as Ramanujan sums, which appear for quadratic forms in an even number of variables.

We now compare various versions of delta symbols in dimension two in applications to a pair of quadratic forms with the this heuristic, that is $R=2$ and $\vec{F}=(F_1,F_2)$ are quadratic forms so that $M=P^2$.

The nested delta symbol of Munshi [27] uses a value of $Q \asymp P^{3/2}$ with a function $p_{q,\vec{a}} = p_q$ that allows an average over $\vec{a}$ with $A \asymp Q^{5/3}$. The value $Q^{5/3}$ is drawn from equation (7) of Munshi's paper, where q_1q_2 different values of $\vec{a}$ are averaged with $q_1 \le P, q_2 \le P^{1/2}$ and the additional condition that q_1 divides a certain quadratic form Q_2. This congruence condition can be detected using additive characters modulo q_1, resulting in $q_1^2q_2$ values of $\vec{a}$, which is typically around $P^{5/2}=Q^{5/3}$ values of $\vec{a}$ as claimed. Therefore, the heuristic suggests that we have an error of size $Q^{s/2}A^{-1/2} \asymp Q^{s/2-5/6}$ which is $P^{(3s-5)/4}$.

Our Theorem 1.2 allows us to take $Q=P^{4/3}$ and to make use of $\vec{a}$-averages with $A \asymp Q$. Thus the heuristic leads to an error term of size $Q^{s/2}A^{-1/2}=P^{2(s-1)/3}$, which is strictly smaller than the heuristic error term $P^{(3s-5)/4}$ from Munshi [27] as soon as $s \ge 8$. It is possible that further savings can be obtained as indicated in Remark 11.1.

Both the nested δ-method of Munshi and our Theorem 1.2 allow a double Kloosterman refinement, that is to make use of averages over q. In Munshi [27], any non-trivial bounds for character sums in the form $\sum_{q_2\le P^{1/2}} \chi(q_2)$ for Dirichlet characters χ with conductors up to P would suffice to handle case for $s=11$. In our case, we need to have more than $Q^{-1/4}$ in savings from the sum $\sum_{q\le Q}\chi(q)$ for Dirichlet characters χ with conductor of size as large as $Q^{s/2}$ to handle the case when $s=9$. Due to the large size of the conductors of the characters, this saving in the q-sum is more difficult to obtain unconditionally compared to that in Munshi [27]. In both methods, when s is even, one has more than square-root cancellations due to Ramanujan sums when the moduli q are generic, although typically this advantage is balanced against worse bounds for non-generic moduli.

According to Heuristic 2, the smaller size of Q should be considerably more advantageous for equations of more variables or higher degrees. These will however bring additional complications by comparison with the quadratic case.

Another point of comparison is the one-dimensional delta symbol method over $\mathbb{Q}(i)$, applied to the complex number $n = F_1 + iF_2$, where F_i are quadratic forms in variable of size P. In the Gaussian integer version of the method, the denominators q would be Gaussian integers $q_1 + iq_2$ of absolute value at most P. However, to apply this to a pair F_1, F_2 with coefficients and variables in $\mathbb{Z}$, we must clear denominators to give rational vectors $\vec{a}/\operatorname{Nm}(q)$, where the denominator now has size P^2. Thus in (12.1) we must actually take $Q = P^2$. We have $A \asymp q^2$, and so $Q^{s/2}A^{-1/2} \asymp P^{s-2}$. However, this heuristic is not accurate because, for sums that come from problems over $\mathbb{Q}[i]$, one should not perform Poisson summation modulo $\operatorname{Nm}(q)$ but rather modulo q. This allows Browning–Pierce–Schindler [8] to obtain an error term smaller than P^{s-4} by this technique for certain pairs of quadratic forms.

Acknowledgments. The authors thank Tim Browning for helpful discussions and providing comments on an earlier version.

Funding. The second author was supported by a Leverhulme Early Career Fellowship, and also received support from ERC grant 101054336.

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