RATIONAL CURVES ON SMOOTH HYPERSURFACES OF LOW DEGREE

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ABSTRACT. We study the family of rational curves on arbitrary smooth hypersurfaces of low degree using tools from analytic number theory.

CONTENTS

1. INTRODUCTION

The geometry of a variety is intimately linked to the geometry of the space of rational curves on it. Given a projective variety X defined over $\mathbb{C}$, a natural object to study is the moduli space of rational curves on X. There are many results in the literature establishing the irreducibility of such mapping spaces, but most statements are only proved for generic X. Following a strategy of Ellenberg and Venkatesh, we shall use tools from analytic number theory to prove such a result for all smooth hypersurfaces of sufficiently low degree.

Let $X \subset \mathbb{P}^n$ be a smooth Fano hypersurface of degree d defined over $\mathbb{C}$, with $n \geqslant 3$. For each positive integer e, the Kontsevich moduli space $\overline{\mathscr{M}}_{0,0}(X,e)$ is a compactification of the space $\mathscr{M}_{0,0}(X,e)$ of morphisms of degree e from $\mathbb{P}^1$ to X, up to isomorphism. According to Kollár [10, Thm. II.1.2/3], any irreducible component of $\overline{\mathscr{M}}_{0,0}(X,e)$ has dimension at least

$$\overline{\mu} = (n+1-d)e + n - 4. \tag{1.1}$$

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Work of Harris, Roth and Starr [7] shows that $\overline{\mathscr{M}}_{0,0}(X,e)$ is an irreducible, local complete intersection scheme of dimension $\overline{\mu}$, provided that X is general and $d<\frac{1}{2}(n+1)$. The restriction on d has since been weakened to $d<\frac{2}{3}(n+1)$ by Beheshti and Kumar [4] (assuming that $n\geqslant 23$), and then to $d\leqslant n-2$ by Riedl and Yang [12].

In the setting $d=3$ of cubic hypersurfaces it is possible to obtain results for all smooth hypersurfaces in the family. Thus Coskun and Starr [5] have shown that $\overline{\mathscr{M}}_{0,0}(X,e)$ is irreducible and of dimension $\overline{\mu}$ for any smooth cubic hypersurface $X\subset\mathbb{P}^n$ over $\mathbb{C}$, provided that $n>4$. (If $n=4$ then $\overline{\mathscr{M}}_{0,0}(X,e)$ has two irreducible components of the expected dimension $\overline{\mu}=2e$.)

At the expense of a much stronger condition on the degree, our main result establishes the irreducibility and dimension of the space $\mathscr{M}_{0,0}(X,e)$, for an arbitrary smooth hypersurface $X\subset\mathbb{P}^n$ over $\mathbb{C}$. Let

$$n_0(d)=2^{d-1}(5d-4). \tag{1.2}$$

We shall prove the following statement.

Theorem 1.1. *Let $X\subset\mathbb{P}^n$ be a smooth hypersurface of degree $d\geqslant 3$ defined over $\mathbb{C}$, with $n\geqslant n_0(d)$. Then for each $e\geqslant 1$ the space $\mathscr{M}_{0,0}(X,e)$ is irreducible and of the expected dimension.*

The example of Fermat hypersurfaces, discussed in [5, §1], shows that the analogous result for $\overline{\mathscr{M}}_{0,0}(X,e)$ is false when $d>3$ and e is large enough. When $e=1$ we have $\overline{\mathscr{M}}_{0,0}(X,1)=\mathscr{M}_{0,0}(X,1)=F_1(X)$, where $F_1(X)$ is the Fano scheme of lines on X. It has been conjectured, independently by Debarre and de Jong, that $\dim F_1(X)=2n-d-3$ for any smooth Fano hypersurface $X\subset\mathbb{P}^n$ of degree d. Beheshti [3] has confirmed this for $d\leqslant 8$. Taking $e=1$ in Theorem 1.1, we conclude that $\dim F_1(X)=2n-d-3$ for any $d\geqslant 3$, provided that $n\geqslant n_0(d)$.

Our proof of Theorem 1.1 ultimately relies on techniques from analytic number theory. The first step is "spreading out", in the sense of Grothendieck [6, §10.4.11] (cf. Serre [13]), which will take us to the analogous problem for smooth hypersurfaces defined over the algebraic closure of a finite field. Passing to a finite field $\mathbb{F}_q$ of sufficiently large cardinality, for a smooth degree d hypersurface $X\subset\mathbb{P}^n_{\mathbb{F}_q}$ defined over $\mathbb{F}_q$, the cardinality of $\mathbb{F}_q$-points on $\mathscr{M}_{0,0}(X,e)$ can be related to the number of $\mathbb{F}_q(t)$-points on X of degree e. We shall access the latter quantity through a function field version of the Hardy–Littlewood circle method. A comparison with the Lang–Weil estimate [8] then allows us to make deductions about the irreducibility and dimension of $\mathscr{M}_{0,0}(X,e)$.

The idea of using the circle method to study the moduli space of rational curves on varieties is due to Ellenberg and Venkatesh. The traditional setting for the circle method is a fixed finite field $\mathbb{F}_q$, with the goal being to understand the $\mathbb{F}_q(t)$-points on X of degree e, as $e \to \infty$. This is the point of view taken in work of Lee [9] on a $\mathbb{F}_q(t)$-version of Birch's work on systems of forms in many variables. In contrast to this, we will be required to handle any fixed $e \geqslant 1$, as $q \to \infty$. Pugin developed an "algebraic circle method" in his 2011 Ph.D. thesis [11] to study the spaces $\mathscr{M}_{0,0}(X, e)$, when $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ is the diagonal cubic hypersurface

$$a_0x_0^3 + \cdots + a_nx_n^3 = 0, \quad (\text{for } a_0, \ldots, a_n \in \mathbb{F}_q^*).$$

Assuming that $n \geqslant 12$ and $\operatorname{char}(\mathbb{F}_q) > 3$, he succeeds in showing that the space $\mathscr{M}_{0,0}(X, e)$ is irreducible and of the expected dimension. Our work extends Pugin's result to arbitrary smooth hypersurfaces of sufficiently low degree, which are defined over the complex numbers.

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2. Spreading out

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d, defined by a homogeneous polynomial

$$F(x_0, \ldots, x_n) = \sum_{\substack{\mathbf{i} \in \mathbb{Z}_{\geqslant 0}^{n+1} \\ i_0 + \cdots + i_n = d}} c_{\mathbf{i}} x_0^{i_0} \ldots x_n^{i_n},$$

with coefficients $c_{\mathbf{i}} \in \mathbb{C}$. Rather than working with $\mathscr{M}_{0,0}(X, 1)$, it will suffice to study the naive space $\operatorname{Mor}_e(\mathbb{P}^1, X)$ of actual maps $\mathbb{P}^1 \to X$ of degree e. The expected dimension of $\operatorname{Mor}_e(\mathbb{P}^1, X)$ is $\mu = \overline{\mu} + 3$, where $\overline{\mu}$ is given by (1.1), since $\mathbb{P}^1$ has automorphism group of dimension 3. We proceed to recall the construction of $\operatorname{Mor}_e(\mathbb{P}^1, X)$.

Let G_e be the set of all homogeneous polynomials in u, v of degree $e \geqslant 1$, with coefficients in $\mathbb{C}$. A *rational curve* of degree e on X is a non-constant morphism $f : \mathbb{P}^1_{\mathbb{C}} \to X$ of degree e. It is given by

$$f = (f_0(u, v), \ldots, f_n(u, v)),$$

with $f_0, \ldots, f_n \in G_e$, with no non-constant common factor in $\mathbb{C}[u, v]$, such that $F(f_0(u, v), \ldots, f_n(u, v))$ vanishes identically. We may regard f as a point in $\mathbb{P}_{\mathbb{C}}^{(n+1)(e+1)-1}$ and the morphisms of degree e on X are parameterised by $\operatorname{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$, which is an open subvariety of $\mathbb{P}_{\mathbb{C}}^{(n+1)(e+1)-1}$ cut out by a system of $de + 1$ equations of degree d. In this way we obtain the expected dimension

$$(n + 1)(e + 1) - 1 - (de + 1) = (n + 1 - d)e + n - 1 = \mu,$$

of $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$. It follows from Kollár [10, Thm. II.1.2] that all irreducible components of $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ have dimension at least μ. In order to establish Theorem 1.1 it will therefore suffice to show that $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ is irreducible, with $\dim \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X) \leqslant \mu$, provided that $n \geqslant n_0(d)$.

The complement to $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ in its closure is the set of $(f_0, \dots, f_n)$ with a common zero. We can obtain explicit equations for $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ by noting that $f_0, \dots, f_n$ have a common zero if and only if the resultant $\mathrm{Res}(\sum_i \lambda_i f_i, \sum_j \mu_j f_j)$ is identically zero as a polynomial in λ_i, μ_j. It is clear that both X and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ are defined by equations with coefficients belonging to the finitely generated $\mathbb{Z}$-algebra $\Lambda = \mathbb{Z}[c_{\mathbf{i}}]$, obtained by adjoining the coefficients of F to $\mathbb{Z}$. In this way we may view X and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ as schemes over Λ, with structure morphisms $X \to \mathrm{Spec}\,\Lambda$ and

$$\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X) \to \mathrm{Spec}\,\Lambda.$$

By Chevalley's upper semicontinuity theorem (see [6, Thm. 13.1.3]), there exists a non-empty open set U of $\mathrm{Spec}\,\Lambda$ such that

$$\dim \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X) \leqslant \dim \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$$

for any closed point $\mathfrak{m} \in U$. Here $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$ denotes the fibre above $\mathfrak{m}$, which is obtained via the base change $\mathrm{Spec}\,\Lambda/\mathfrak{m} \to \mathrm{Spec}\,\Lambda$. Likewise, since integrality is an open condition, the space $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)$ will be irreducible if $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$ is.

Choose a maximal ideal $\mathfrak{m}$ in U. The quotient $\Lambda/\mathfrak{m}$ is a finite field by arithmetic weak Nullstellensatz. By enlarging Λ, we may assume that it contains $1/d!$. In particular, it follows that $\mathrm{char}(\Lambda/\mathfrak{m}) = p$, say, with $p > d$, since any prime less than or equal to d is invertible in Λ. The quasi-projective varieties $X_{\mathfrak{m}}$ and $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{C}}, X)_{\mathfrak{m}}$ are defined over $\overline{\mathbb{F}}_p$, being given explicitly by reducing modulo $\mathfrak{m}$ the coefficients of the original system of defining equations. By further enlarging Λ, if necessary, we may assume that $X_{\mathfrak{m}}$ is smooth. There exists a finite field $\mathbb{F}_{q_0}$ such that $X_{\mathfrak{m}}$ and $\mathrm{Mor}_e(\mathbb{P}^1, X_{\mathbb{C}})_{\mathfrak{m}}$ are both defined over $\mathbb{F}_{q_0}$. In view of the Lang–Weil estimate, Theorem 1.1 is a direct consequence of the following result.

Theorem 2.1. *Let $n \geqslant n_0(d)$ and let $X \subset \mathbb{P}^n_{\mathbb{F}_q}$ be a smooth hypersurface of degree $d \geqslant 3$ defined over a finite field $\mathbb{F}_q$, with $\mathrm{char}(\mathbb{F}_q) > d$. Then for each $e \geqslant 1$ we have*

$$\lim_{\ell \to \infty} q^{-\ell\mu} \# \mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{F}_q}, X)(\mathbb{F}_{q^\ell}) \leqslant 1.$$

3. The Hardy–Littlewood circle method

We now initiate the proof of Theorem 2.1. We henceforth redefine q^ℓ to be q and we replace n by $n-1$ in the statement of the theorem. In particular the expected dimension is now $\mu = (n-d)e + n - 2$. Our proof of Theorem 2.1 is based on a version of the Hardy–Littlewood circle method for the function field $K = \mathbb{F}_q(t)$, always under the assumption that $\mathrm{char}(\mathbb{F}_q) > d$. The main input for this comes from work of Lee [9], combined with our own recent contribution to the subject, in the setting of cubic forms [2].

We begin by laying down some basic notation and terminology. To begin with, for any real number R we set $\widehat{R} = q^R$. Let $\mathscr{O} = \mathbb{F}_q[t]$ be the ring of integers of K and let Ω be the set of places of K. These correspond to either monic irreducible polynomials ϖ in $\mathscr{O}$, which we call the *finite primes*, or the *prime at infinity* t^{-1} which we usually denote by ∞. The associated absolute value $|\cdot|_v$ is either $|\cdot|_\varpi$ for some prime $\varpi \in \mathscr{O}$ or $|\cdot|$, according to whether v is a finite or infinite place, respectively. These are given by

$$|a/b|_\varpi = \left(\frac{1}{q^{\deg \varpi}}\right)^{\mathrm{ord}_\varpi(a/b)} \qquad \text{and} \qquad |a/b| = q^{\deg a - \deg b},$$

for any $a/b \in K^*$. We extend these definitions to K by taking $|0|_\varpi = |0| = 0$.

For $v \in \Omega$ we let K_v denote the completion of K at v with respect to $|\cdot|_v$. We may identify K_∞ with the set

$$\mathbb{F}_q((1/t)) = \left\{ \sum_{i \leqslant N} a_i t^i : \text{for } a_i \in \mathbb{F}_q \text{ and some } N \in \mathbb{Z} \right\}.$$

We can extend the absolute value at the infinite place to K_∞ to get a non-archimedean absolute value $|\cdot| : K_\infty \to \mathbb{R}_{\geqslant 0}$ given by $|\alpha| = q^{\mathrm{ord}\,\alpha}$, where $\mathrm{ord}\,\alpha$ is the largest $i \in \mathbb{Z}$ such that $a_i \neq 0$ in the representation $\alpha = \sum_{i \leqslant N} a_i t^i$. In this context we adopt the convention $\mathrm{ord}\, 0 = -\infty$ and $|0| = 0$. We extend this to vectors by setting $|\mathbf{x}| = \max_{1 \leqslant i \leqslant n} |x_i|$, for any $\mathbf{x} \in K_\infty^n$.

Next, we put

$$\mathbb{T} = \{\alpha \in K_\infty : |\alpha| < 1\} = \left\{ \sum_{i \leqslant -1} a_i t^i : \text{for } a_i \in \mathbb{F}_q \right\}.$$

Since $\mathbb{T}$ is a locally compact additive subgroup of K_∞ it possesses a unique Haar measure $\mathrm{d}\alpha$, which is normalised so that $\int_{\mathbb{T}} \mathrm{d}\alpha = 1$. We can extend $\mathrm{d}\alpha$ to a (unique) translation-invariant measure on K_∞ in such a way that

$$\int_{\{\alpha \in K_\infty : |\alpha| < \widehat{N}\}} \mathrm{d}\alpha = \widehat{N},$$

for any $N \in \mathbb{Z}_{>0}$. These measures also extend to $\mathbb{T}^n$ and K_∞^n, for any $n \in \mathbb{Z}_{>0}$. There is a non-trivial additive character $e_q : \mathbb{F}_q \to \mathbb{C}^*$ defined for each $a \in \mathbb{F}_q$ by taking $e_q(a) = \exp(2\pi i\, \mathrm{Tr}(a)/p)$, where $\mathrm{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ denotes the trace map. This character induces a non-trivial (unitary) additive character $\psi : K_\infty \to \mathbb{C}^*$ by defining $\psi(\alpha) = e_q(a_{-1})$ for any $\alpha = \sum_{i \leqslant N} a_i t^i$ in K_∞.

Let $F \in \mathbb{F}_q[\mathbf{x}]$ be a non-singular form of degree $d \geqslant 3$, with $\mathbf{x} = (x_1, \dots, x_n)$. We may express this polynomial as

$$F(\mathbf{x}) = \sum_{i_1,\dots,i_d=1}^{n} c_{i_1,\dots,i_d} x_{i_1} \dots x_{i_d},$$

with coefficients $c_{i_1,\dots,i_d} \in \mathbb{F}_q$. In particular the height H_F and discriminant Δ_F of F satisfy $H_F = \max_{\mathbf{i}} |c_{\mathbf{i}}| = 1$ and $|\Delta_F| = 1$. We will make frequent use of these facts in what follows. Associated to F are the multilinear forms

$$\Psi_i(\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(d-1)}) = \sum_{i_1,\dots,i_{d-1}=1}^{n} c_{i_1,\dots,i_{d-1},i} x_{i_1}^{(1)} \dots x_{i_{d-1}}^{(d-1)}, \tag{3.1}$$

for $1 \leqslant i \leqslant n$.

To establish Theorem 2.1 we work with the naive space

$$M_e = \{\mathbf{x} = (x_1, \dots, x_n) \in G_e(\mathbb{F}_q)^n \setminus \{\mathbf{0}\} : F(\mathbf{x}) = 0\},$$

where $G_e(\mathbb{F}_q)$ is the set of binary forms of degree e with coefficients in $\mathbb{F}_q$. Thus M_e corresponds to the $\mathbb{F}_q$-points on the affine cone of $\mathrm{Mor}_e(\mathbb{P}^1_{\mathbb{F}_q}, X)$. Let us set

$$\widehat{\mu} = \mu + 1 = (n-d)e + n - 1 = (e+1)n - de - 1. \tag{3.2}$$

It will clearly suffice to show that

$$\lim_{q\to\infty} q^{-\widehat{\mu}} \# M_e \leqslant 1, \tag{3.3}$$

for $n > n_0(d)$, where $n_0(d)$ is given by (1.2). We proceed by relating the counting function $\# M_e$ to the counting function that lies at the heart of our earlier investigation [2].

Let $w : K_\infty^n \to \{0,1\}$ be given by $w(\mathbf{x}) = \prod_{1\leqslant i\leqslant n} w_\infty(x_i)$, where

$$w_\infty(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Putting $P = t^{e+1}$, we then have $\# M_e \leqslant N(P)$, where

$$N(P) = \sum_{\substack{\mathbf{x}\in\mathscr{O}^n \\ F(\mathbf{x})=0}} w(\mathbf{x}/P).$$

It follows from [2, Eq. (4.1)] that for any $Q \geqslant 1$ we have

$$N(P) = \sum_{\substack{r\in\mathscr{O} \\ |r|\leqslant\widehat{Q} \\ r \text{ monic}}} \sideset{}{^*}\sum_{|a|<|r|} \int_{|\theta|<|r|^{-1}\widehat{Q}^{-1}} S\left(\frac{a}{r} + \theta\right) \mathrm{d}\theta, \tag{3.4}$$

where $\sum^*$ means that the sum is taken over residue classes $|a| < |r|$ for which $\gcd(a,r) = 1$, and where

$$S(\alpha) = \sum_{\mathbf{x}\in\mathscr{O}^n} \psi(\alpha F(\mathbf{x}))w(\mathbf{x}/P), \tag{3.5}$$

for any $\alpha \in \mathbb{T}$. We will work with the choice $Q = d(e+1)/2$, so that $\widehat{Q} = |P|^{d/2}$.

The *major arcs* for our problem are given by $r = 1$ and $|\theta| < |P|^{-d}q^{d-1}$. We let the *minor arcs* be everything else: i.e. those $\alpha = a/r+\theta$ appearing in (3.4) for which either $|r| > q$, or else $r = 1$ and $|\theta| \geqslant |P|^{-d}q^{d-1}$. The contribution $N_{\mathrm{major}}(P)$ from the major arcs is easy to deal with. Indeed, for $|\theta| < |P|^{-d}q^{d-1}$ and $|\mathbf{x}| < |P|$ we have $|\theta F(\mathbf{x})| < |P|^{-d}q^{d-1}q^{de} = q^{-1}$, whence $\psi(\theta F(\mathbf{x})) = 1$. Thus $S(\alpha) = |P|^n$, for $\alpha = \theta$ belonging to the major arcs, and so

$$N_{\mathrm{major}}(P) = |P|^n \int_{|\theta|<|P|^{-d}q^{d-1}} \mathrm{d}\theta = |P|^{n-d}q^{d-1} = q^{\widehat{\mu}}.$$

In order to prove (3.3), it therefore remains to show that

$$\lim_{q\to\infty} q^{-\widehat{\mu}} N_{\mathrm{minor}}(P) = 0, \tag{3.6}$$

for $n > n_0(d)$, where $N_{\mathrm{minor}}(P)$ is the overall contribution to (3.4) from the minor arcs. This will complete the proof of Theorem 2.1.

4. The geometry of numbers

The purpose of this section is to record a technical result about lattice point counting over K_∞. A *lattice* in K_∞^N is a set of points of the form $\mathbf{x} = \Lambda\mathbf{u}$, where Λ is a $N \times N$ matrix over K_∞ and $\mathbf{u}$ runs over elements of $\mathscr{O}^N$. By an abuse of notation we will also denote the set of such points by Λ. Given a lattice M, the *adjoint lattice* Λ is defined to satisfy $\Lambda^T M = I_N$, where I_N is the $N \times N$ identity matrix.

Let $\gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in K_∞. Given any positive integer m, we define the special lattice

$$M_m = \begin{pmatrix} t^{-m}I_n & 0 \\ t^m\gamma & t^m I_n \end{pmatrix},$$

with corresponding adjoint lattice

$$\Lambda_m = \begin{pmatrix} t^m I_n & -t^m\gamma \\ 0 & t^{-m}I_n \end{pmatrix}.$$

Let $\{\widehat{R}_1, ..., \widehat{R}_{2n}\}$ denote the successive minima of the lattice corresponding to M_m. For any vector $\mathbf{x} \in K_\infty^{2n}$ let $\mathbf{x}_1 = (x_1, \dots, x_n)$ and $\mathbf{x}_2 = (x_{n+1}, \dots, x_{2n})$. We claim that M_m and Λ_m can be identified with one another. Now M_m is the set of points $\mathbf{x} = M_m\mathbf{u}$ where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ runs over elements of $\mathscr{O}^{2n}$. Likewise, Λ_m is the set of points $\mathbf{y} = \Lambda_m\mathbf{v}$ where $\mathbf{v} = (\mathbf{u}_1, \mathbf{v}_2)$ runs over elements of $\mathscr{O}^{2n}$. We can therefore identify M_m with Λ_m through the process of changing the sign of $\mathbf{v}_2$, then the sign of $\mathbf{y}_2$, then switching $\mathbf{v}_1$ with $\mathbf{v}_2$, and finally interchanging $\mathbf{y}_1$ and $\mathbf{y}_2$. It now follows from [9, Lemma B.6] that

$$R_\nu + R_{2n-\nu+1} = 0, \tag{4.1}$$

for $1 \leqslant \nu \leqslant n$. Note that an important step in the proof of [9, Lemma B.6] is a non-archimedean version of Gram–Schmidt orthogonalisation, which is used without reference in the proof of [9, Lemma B.3]. This deficit is remedied by appealing to recent work of Usher and Zhang [14, Theorem 2.16].

For any $Z \in \mathbb{R}$ and any lattice Γ we define the counting function

$$\Gamma(Z) = \#\{\mathbf{x} \in \Gamma : |\mathbf{x}| < \widehat{Z}\}.$$

Note that $\Gamma(Z) = \Gamma(\lceil Z \rceil)$ for any $Z \in \mathbb{R}$. We proceed to establish the following inequality.

Lemma 4.1. *Let $m, Z_1, Z_2 \in \mathbb{Z}$ such that $Z_1 \leqslant Z_2 \leqslant 0$. Then we have*

$$\frac{M_m(Z_1)}{M_m(Z_2)} \geqslant \left(\frac{\widehat{Z}_1}{\widehat{Z}_2}\right)^n.$$

Proof. Let $1 \leqslant \mu, \nu \leqslant 2n$ be such that $R_\mu < Z_1 \leqslant R_{\mu+1}$ and $R_\nu < Z_2 \leqslant R_{\nu+1}$. Since R_j is a non-decreasing sequence which satisfies $R_j + R_{2n-j+1} = 0$, by (4.1), we must have $0 \leqslant R_{n+1}$, whence in fact $\mu \leqslant \nu \leqslant n$. It follows from [9, Lemma B.5] that

$$\frac{M_m(Z_1)}{M_m(Z_2)} = \begin{cases} 1 & \text{if } Z_1, Z_2 < R_1, \\ \left(\prod_{j=1}^{\nu} \widehat{R}_j/\widehat{Z_1}\right)(\widehat{Z_1}/\widehat{Z_2})^\nu & \text{if } Z_1 < R_1 \leqslant Z_2, \\ \left(\prod_{j=\mu+1}^{\nu} \widehat{R}_j/\widehat{Z_1}\right)(\widehat{Z_1}/\widehat{Z_2})^\nu & \text{if } R_1 \leqslant Z_1 \leqslant Z_2, \end{cases}$$

The statement of the lemma is now obvious. □

As above, let $\gamma = (\gamma_{ij})$ be a symmetric $n \times n$ matrix with entries in K_∞. For $1 \leqslant i \leqslant n$ we introduce the linear forms

$$L_i(u_1, \dots, u_n) = \sum_{j=1}^{n} \gamma_{ij} u_j.$$

Next, for given real numbers a, Z, we let $N(a, Z)$ denote the number of vectors $(u_1, \dots, u_{2n}) \in \mathscr{O}^{2n}$ such that

$$|u_j| < \widehat{a}\widehat{Z} \quad \text{and} \quad |L_j(u_1, \dots, u_n) + u_{j+n}| < \frac{\widehat{Z}}{\widehat{a}} \quad \text{for } 1 \leqslant j \leqslant n.$$

If we put $m = \lfloor a \rfloor$, then it is clear that

$$M_m(Z - \{a\}) \leqslant N(a, Z) \leqslant M_m(Z + \{a\}),$$

where $\{a\}$ denotes the fractional part of a. The following result is a direct consequence of Lemma 4.1.

Lemma 4.2. *Let $a, Z_1, Z_2 \in \mathbb{R}$ with $Z_1 \leqslant Z_2 \leqslant 0$. Then we have*

$$\frac{N(a, Z_1)}{N(a, Z_2)} \geqslant \widehat{K}^n,$$

where $K = \lceil Z_1 - \{a\} \rceil - \lceil Z_2 + \{a\} \rceil$.

5. Weyl differencing

In everything that follows we shall assume that $\mathrm{char}(\mathbb{F}_q) > d$ and we will allow all our implied constants to depend on d and n. Any dependence on q will be made completely explicit. This section is concerned with a careful analysis of the exponential sum (3.5), using the function field version of Weyl differencing that was worked out by Lee [9]. Our task is to make the dependence on q completely explicit and it turns out that gaining satisfactory control requires considerable care. Since we are concerned with hypersurfaces one needs to take $R = 1$ in [9, §3].

For any $\beta = \sum_{i \leqslant N} b_i t^i \in K_\infty$, we let $\|\beta\| = |\sum_{i \leqslant -1} b_i t^i|$. Recalling the definition (3.1) of the multilinear forms associated to F, we let

$$N(\alpha) = \#\left\{ \underline{\mathbf{u}} \in \mathscr{O}^{(d-1)n} : \begin{array}{l} |\mathbf{u}_1|, \dots, |\mathbf{u}_{d-1}| < |P| \\ \|\alpha \Psi_i(\underline{\mathbf{u}})\| < |P|^{-1} \ (\forall i \leqslant n) \end{array} \right\},$$

where $\underline{\mathbf{u}} = (\mathbf{u}_1, \dots, \mathbf{u}_{d-1})$. We begin with an application of [9, Cor. 3.3], which leads to the inequality

$$|S(\alpha)|^{2^{d-1}} \leqslant |P|^{(2^{d-1}-d+1)n} N(\alpha), \tag{5.1}$$

for any $\alpha \in \mathbb{T}$.

The next stage in the analysis of $S(\alpha)$ is a multiple application of the function field analogue of Davenport's "shrinking lemma", as proved in [9, Lemma 3.4], ultimately leading to [9, Lemma 3.5]. Unfortunately the implied constant in these estimates is allowed to depend on q and so we must work harder to control it. Let

$$N_\eta(\alpha) = \#\left\{ \underline{\mathbf{u}} \in \mathscr{O}^{(d-1)n} : \begin{array}{l} |\mathbf{u}_1|, \dots, |\mathbf{u}_{d-1}| < |P|^{\eta} \\ \|\alpha \Psi_i(\underline{\mathbf{u}})\| < |P|^{-d+(d-1)\eta} \ (\forall i \leqslant n) \end{array} \right\},$$

for any parameter $\eta \in [0, 1]$. Recalling that $P = t^{e+1}$, we shall prove the following uniform version of [9, Lemma 3.5].

Lemma 5.1. *Let $\alpha \in \mathbb{T}$ and suppose that $\eta \in [0,1)$ is chosen so that*

$$\frac{(e+1)(\eta+1)}{2} \in \mathbb{Z}. \tag{5.2}$$

Then we have

$$|S(\alpha)|^{2^{d-1}} \leqslant \frac{|P|^{2^{d-1}n}}{|P|^{\eta(d-1)n}} N_\eta(\alpha).$$

Proof. In view of (5.1), in order to establish the result we need to prove that

$$N_\eta(\alpha) \geqslant (|P|^{-n+\eta n})^{d-1} N(\alpha),$$

for any $\eta \in [0,1)$ satisfying the hypothesis (5.2). For each $v \in \{0, \dots, d-1\}$, define $N^{(v)}(\alpha)$ to be the number of $\underline{\mathbf{u}} \in \mathscr{O}^{(d-1)n}$ such that

$$|\mathbf{u}_1|, \dots, |\mathbf{u}_v| < |P|^\eta, \quad |\mathbf{u}_{v+1}|, \dots, |\mathbf{u}_{d-1}| < |P| \tag{5.3}$$

and $\|\alpha \Psi_i(\underline{\mathbf{u}})\| < |P|^{-v-1+v\eta}$, for $1 \leqslant i \leqslant n$. Thus we have $N^{(0)}(\alpha) = N(\alpha)$ and $N^{(d-1)}(\alpha) = N_\eta(\alpha)$. It will suffice to show that

$$N^{(v)}(\alpha) \geqslant |P|^{-n+\eta n} N^{(v-1)}(\alpha),$$

for each $v \in \{1, \dots, d-1\}$.

Fix a choice of v, together with $\mathbf{u}_1, \dots, \mathbf{u}_{v-1}, \mathbf{u}_{v+1}, \dots, \mathbf{u}_{d-1} \in \mathscr{O}^n$ such that (5.3) holds. For each $1 \leqslant i \leqslant n$ we consider the linear form

$$L_i(\mathbf{u}) = \alpha \Psi_i(\mathbf{u}_1, \dots, \mathbf{u}_{v-1}, \mathbf{u}, \mathbf{u}_{v+1}, \dots, \mathbf{u}_{d-1}) = \sum_{j=1}^n \gamma_{ij} u_j,$$

say, for a suitable symmetric $n \times n$ matrix $\gamma = (\gamma_{ij})$, with entries in K_∞. Given real numbers a and Z, define $N(a, Z)$ to be the number of vectors $(u_1, \dots, u_{2n})$ in $\mathscr{O}^{2n}$ satisfying

$$|u_j| < \widehat{Z+a} \quad \text{and} \quad |L_j(u_1, \dots, u_n) - u_{j+n}| < \widehat{Z-a}, \quad \text{for } 1 \leqslant j \leqslant n.$$

We are interested in estimating the number of $\mathbf{u} \in \mathscr{O}^n$ such that $|\mathbf{u}| < |P|^\eta$ and $\|L_i(\mathbf{u})\| < |P|^{-v-1+v\eta}$, for $1 \leqslant i \leqslant n$, in terms of the number of $\mathbf{u} \in \mathscr{O}^n$ such that $|\mathbf{u}| < |P|$ and $\|L_i(\mathbf{u})\| < |P|^{-v+(v-1)\eta}$, for $1 \leqslant i \leqslant n$. That is, we wish to compare $N(a, Z_1)$ with $N(a, Z_2)$, where

$$\widehat{a} = |P|^{(v+1-(v-1)\eta)/2}, \quad \widehat{Z_1} = |P|^{(v+1)(\eta-1)/2}, \quad \widehat{Z_2} = |P|^{(v-1)(\eta-1)/2}.$$

Note that $\widehat{a}\widehat{Z}_1 = |P|^\eta$ and $\widehat{a}\widehat{Z}_2 = |P|$. Moreover, our hypothesis (5.2) implies that

$$a = \frac{(e+1)(v+1)}{2} - \frac{(v-1)(e+1)\eta}{2} = v(e+1) - \frac{(v-1)(e+1)(\eta+1)}{2} \in \mathbb{Z}.$$

Similarly, (5.2) implies that $Z_1, Z_2 \in \mathbb{Z}$. It now follows from Lemma 4.2 that

$$\frac{N(a,Z_1)}{N(a,Z_2)} \geqslant \left(\widehat{Z_1 - Z_2}\right)^n = |P|^{-n+\eta n},$$

which thereby completes the proof of the lemma. □

Lemma 5.1 doesn't suffice to handle the case $e = 1$ of lines on the hypersurface. To circumvent this difficulty we shall invoke a simpler version of the shrinking lemma, as follows.

Lemma 5.2. *Let $\alpha \in \mathbb{T}$ and let $v \in \{1, \dots, d\}$. Then we have*

$$|S(\alpha)|^{2^{d-1}} \leqslant |P|^{(2^{d-1}-d+1)n} q^{e(v-1)n} M^{(v)}(\alpha),$$

where $M^{(v)}(\alpha)$ is the number of $\underline{\mathbf{u}} \in \mathscr{O}^{(d-1)n}$ such that

$$|\mathbf{u}_1|, \dots, |\mathbf{u}_{v-1}| < q, \quad |\mathbf{u}_v|, \dots, |\mathbf{u}_{d-1}| < |P|.$$

and $\|\alpha \Psi_i(\underline{\mathbf{u}})\| < |P|^{-1}$ for $1 \leqslant i \leqslant n$.

Proof. Noting that $N(\alpha) = M^{(1)}(\alpha)$, it follows from (5.1) that it will be enough to prove that $M^{(v-1)}(\alpha) \leqslant q^{en} M^{(v)}(\alpha)$ for $2 \leqslant v \leqslant d$. The proof follows that of Lemma 5.1 and so we shall be brief. Let $\mathbf{u}_1, \dots, \mathbf{u}_{v-1}, \mathbf{u}_{v+1}, \dots, \mathbf{u}_{d-1} \in \mathscr{O}^n$ be vectors satisfying

$$|\mathbf{u}_1|, \dots, |\mathbf{u}_{v-1}| < q, \quad |\mathbf{u}_{v+1}|, \dots, |\mathbf{u}_{d-1}| < |P|.$$

Let γ and $N(a, Z)$ be as in the proof of Lemma 5.1, corresponding to this choice. Lemma 4.2 clearly implies that

$$\frac{N(e+1,-e)}{N(e+1,0)} \geqslant q^{-en}.$$

However, $N(e+1,-e)$ denotes the number of $\mathbf{u} \in \mathscr{O}^n$ such that $|\mathbf{u}| < q$ and $\|L_i(\mathbf{u})\| < q^{-2e-1}$, for $1 \leqslant i \leqslant n$. The lemma follows on noting that $q^{-2e-1} < q^{-e-1} = |P|^{-1}$. □

The next step is an application of the function field analogue of Heath-Brown's Diophantine approximation lemma, as worked out in [9, Lemma 3.6]. Let $\alpha = a/r + \theta$, where $a/r \in K$ and $\theta \in \mathbb{T}$. Note that the maximum absolute value of the coefficients of each multilinear form Ψ_j is 1. We shall apply [9, Lemma 3.6] with $\widehat{M} = |P|^{(d-1)\eta}$ and $\widehat{Y} = |P|^{d-(d-1)\eta}$. We desire a maximal choice of $\eta \geqslant 0$ such that

$$|P|^{(d-1)\eta} < \min\left\{|P|^{d-1}, \frac{1}{|r\theta|}, \frac{|P|^d}{|r|}\right\}$$

and

$$|P|^{(d-1)\eta} \leqslant |r| \max\left\{1, |P^d\theta|\right\}.$$

This leads to the constraint $(e+1)\eta \leqslant \Gamma$, where

$$\Gamma = \frac{1}{d-1} \operatorname{ord}\left(\min\left\{\frac{|P|^{d-1}}{q}, \frac{1}{q|r\theta|}, \frac{|P|^d}{q|r|}, |r|\max\left\{1, |P^d\theta|\right\}\right\}\right), \tag{5.4}$$

in which we abuse notation and denote by ord the integer exponent of q that appears. For $i \in \{0,1\}$, we let $[\Gamma]_i$ denote the largest non-negative integer not exceeding Γ, which is congruent to i modulo 2. We then choose our parameter η via

$$(e+1)\eta = \begin{cases} [\Gamma]_0 & \text{if } 2 \nmid e, \\ [\Gamma]_1 & \text{if } 2 \mid e. \end{cases} \tag{5.5}$$

One notes that $(e+1)\eta \leqslant \Gamma$ and, furthermore, that (5.2) is satisfied.

It now follows from [9, Lemma 3.6] that $N_\eta(\alpha) \leqslant U_\eta$, where U_η denotes the number of $\underline{\mathbf{u}} \in \mathscr{O}^{(d-1)n}$ such that $|\mathbf{u}_1|, \dots, |\mathbf{u}_{d-1}| < |P|^\eta$ and

$$\Psi_i(\underline{\mathbf{u}}) = 0, \quad \text{for } 1 \leqslant i \leqslant n.$$

The calculation in [9, §3] shows that the latter system of equations defines an affine variety of dimension at most $(d-2)n$. We now apply [2, Lemma 2.8]. Since $|P|^\eta = q^{(e+1)\eta}$, with $(e+1)\eta \in \mathbb{Z}$, this directly yields the existence of a positive constant $c_{d,n}$, independent of q, such that $U_\eta \leqslant c_{d,n}|P|^{\eta(d-2)n}$. Inserting this into Lemma 5.1, we therefore arrive at the following conclusion.

Lemma 5.3. *Let $L = 2^{-d+1}n$, let $a/r \in K$ and let $\theta \in \mathbb{T}$. Then there exists a constant $c_{d,n} > 0$, independent of q, such that*

$$|S(a/r+\theta)| \leqslant c_{d,n}|P|^{n-L\eta},$$

where η is given by (5.5).

It turns out that this estimate is inefficient when $|r|$ is small. Let

$$\kappa = \begin{cases} 1 & \text{if } 2 \nmid e, \\ 0 & \text{if } 2 \mid e. \end{cases} \tag{5.6}$$

It will also be advantageous to consider the effect of taking $(e+1)\eta = 1+\kappa$, instead of (5.5). Since

$$\frac{(e+1)(\eta+1)}{2} = 1 + \frac{e+\kappa}{2} \in \mathbb{Z},$$

it follows from Lemma 5.1 that

$$|S(\alpha)| \leqslant \frac{|P|^n \mathscr{N}^{2^{-d+1}}}{q^{(1+\kappa)(d-1)L}}, \tag{5.7}$$

where

$$\mathscr{N} = \#\left\{ \underline{\mathbf{u}} \in \mathscr{O}^{(d-1)n} : \begin{array}{l} |\mathbf{u}_1|, \dots, |\mathbf{u}_{d-1}| \leqslant q^{\kappa} \\ \|\alpha \Psi_i(\underline{\mathbf{u}})\| < q^{\kappa(d-1)-de-1} \ (\forall i \leqslant n) \end{array} \right\}, \tag{5.8}$$

Supposing that $\alpha = a/r + \theta$ for $a/r \in K$ and $\theta \in \mathbb{T}$, our argument now bifurcates according to the size of $|r|$.

Lemma 5.4. *Let $L = 2^{-d+1}n$, let $a/r \in K$ and let $\theta \in \mathbb{T}$. Assume that*

(i) *$e \geqslant 1$, $q \leqslant |r| < q^{de+1-\kappa(d-1)}$ and $|r\theta| < q^{-\kappa(d-1)}$; or*
(ii) *$e = 1$, $q^2 \leqslant |r| \leqslant q^d$ and $|r\theta| \leqslant q^{-d}$.*

Then there exists a constant $c'_{d,n} > 0$, independent of q, such that

$$|S(a/r + \theta)| \leqslant c'_{d,n} |P|^n q^{-L}.$$

Proof. To deal with case (i) we apply [9, Lemma 3.6] with $Y = de+1-\kappa(d-1)$ and $M = \kappa(d-1) + \frac{1}{2}$. Our hypotheses ensure that $|r| < \widehat{Y}$ and $|r\theta| < \widehat{M}^{-1}$. Thus it follows that $\Psi_i(\underline{\mathbf{u}}) \equiv 0 \bmod r$ in (5.8), for all $i \leqslant n$. In particular we have $\mathscr{N} = 0$ unless $\kappa = 1$, which we now assume.

Pick a prime $\varpi \mid r$ with $|\varpi| \geqslant q$. If $|\varpi| \leqslant q^2$ we may break into residue classes modulo ϖ, finding that

$$\mathscr{N} \leqslant \sum_{\mathbf{v}_1, \dots, \mathbf{v}_{d-1}} \# \left\{ |\mathbf{u}_1|, \dots, |\mathbf{u}_{d-1}| \leqslant q : \mathbf{u}_i \equiv \mathbf{v}_i \bmod \varpi, \text{ for } 1 \leqslant i \leqslant d-1 \right\},$$

where the sum is over all $\underline{\mathbf{v}} = (\mathbf{v}_1, \dots, \mathbf{v}_{d-1}) \in \mathbb{F}_{\varpi}^{(d-1)n}$ such that $\Psi_i(\underline{\mathbf{v}}) = 0$, for all $i \leqslant n$, over $\mathbb{F}_{\varpi}$. The inner cardinality is $O((q^2/|\varpi|)^{(d-1)n})$, with an implied constant that is independent of q. We may use the Lang–Weil estimate to deduce that the outer sum is $O(|\varpi|^{(d-2)n})$, again with an implied constant that depends at most on d and n. Hence we get the overall contribution

$$\mathscr{N} \ll \frac{q^{2(d-1)n}}{|\varpi|^n} \leqslant q^{2(d-1)n-n}.$$

Alternatively, if $|\varpi| > q^2$, we may assume that the system of equations $\Psi_i = 0$, for $i \leqslant n$, has dimension $(d-2)n$ over $\mathbb{F}_{\varpi}$. We now appeal to an argument of Browning and Heath-Brown [1, Lemma 4]. Using induction on the dimension, as in the proof of [1, Eq. (3.7)], we easily conclude that

$$\mathscr{N} \ll (q^2)^{(d-2)n} \leqslant q^{2(d-1)n-2n},$$

for an implied constant that only depends on d and n. Recalling that $\kappa = 1$, the first part of the lemma now follows on substituting these bounds into (5.7).

We now consider case (ii), in which $e = 1$, $q^2 \leqslant |r| \leqslant q^d$ and $|r\theta| \leqslant q^{-d}$. Let $|a/r| = q^{-\alpha}$ for $1 \leqslant \alpha \leqslant d$. Let $v \in \{1, \dots, d\}$ be such that $d - v - \alpha = -1$.

Then an application of Lemma 5.2 yields

$$\begin{aligned}|S(\alpha)|^{2^{d-1}} &\leqslant |P|^{(2^{d-1}-d+1)n} q^{(v-1)n} M^{(v)}(\alpha)\\ &= |P|^{2^{d-1}n} \cdot q^{(-2d+1+v)n} M^{(v)}(\alpha).\end{aligned}$$

Let $\underline{\mathbf{u}} \in \mathscr{O}^{n(d-1)}$ be counted by $M^{(v)}(\alpha)$. Since $|\theta| \leqslant q^{-d-2}$, it follows that $|\theta\Psi_i(\underline{\mathbf{u}})| \leqslant q^{-d-2} \cdot q^{d-v} = q^{-2-v} \leqslant q^{-3}$, for $1 \leqslant i \leqslant n$. Similarly, for $1 \leqslant i \leqslant n$, we have $|\frac{a}{r}\Psi_i(\underline{\mathbf{u}})| \leqslant q^{-\alpha} \cdot q^{d-v} = q^{-1}$. If we write $\mathbf{u}_j = \mathbf{u}_j' + t\mathbf{u}_j''$, for $v \leqslant j \leqslant d$, where $\mathbf{u}_j', \mathbf{u}_j'' \in \mathbb{F}_q^n$, then the coefficient of t^{-1} in the t-expansion of $\frac{a}{r}\Psi_i(\underline{\mathbf{u}})$ is equal to $\Psi_i(\mathbf{u}_1, \dots, \mathbf{u}_{v-1}, \mathbf{u}_v'', \dots, \mathbf{u}_{d-1}'')$. The condition $\|\alpha\Psi_i(\underline{\mathbf{u}})\| < |P|^{-1}$ in $M^{(v)}(\alpha)$ implies that this coefficient must necessarily vanish, whence $M^{(v)}(\alpha)$ is at most the number of $\mathbf{u}_1, \dots \mathbf{u}_{v-1}, \mathbf{u}_v', \dots, \mathbf{u}_{d-1}', \mathbf{u}_v'', \dots, \mathbf{u}_{d-1}'' \in \mathbb{F}_q^n$ for which $\Psi_i(\mathbf{u}_1, \dots, \mathbf{u}_{v-1}, \mathbf{u}_v'', \dots, \mathbf{u}_{d-1}'') = 0$, for $1 \leqslant i \leqslant n$. Thus

$$M^{(v)}(\alpha) \ll q^{(d-v)n} \cdot q^{(d-2)n} = q^{(2d-v-2)n},$$

by the Lang–Weil estimate, which implies the statement of the lemma. □

Lemma 5.5. *Let $L = 2^{-d+1}n$ and let $\theta \in \mathbb{T}$. Assume that*

$$q^{-de-1} \leqslant |\theta| \leqslant q^{-1-\kappa(d-1)}.$$

Then there exists a constant $c_{d,n}'' > 0$, independent of q, such that

$$|S(\theta)| \leqslant c_{d,n}'' |P|^n q^{-L}.$$

Proof. The upper bound assumed of $|\theta|$ implies that $|\theta\Psi_i(\underline{\mathbf{u}})| \leqslant q^{-1}$ in (5.8), for $1 \leqslant i \leqslant n$. Hence $\|\theta\Psi_i(\underline{\mathbf{u}})\| = |\theta\Psi_i(\underline{\mathbf{u}})|$ for $1 \leqslant i \leqslant n$. Since $\alpha = \theta$ and $|\theta| \geqslant q^{-de-1}$, it follows that the condition $\|\alpha\Psi_i(\underline{\mathbf{u}})\| < q^{\kappa(d-1)-de-1}$ is equivalent to $|\Psi_i(\underline{\mathbf{u}})| < q^{\kappa(d-1)}$. If $\kappa = 0$ then it follows from (5.8) that

$$\mathscr{N} = \#\left\{\underline{\mathbf{u}} \in \mathbb{F}_q^{(d-1)n} : \Psi_i(\underline{\mathbf{u}}) = 0 \ (\forall i \leqslant n)\right\} \ll q^{(d-2)n},$$

by the Lang–Weil estimate. If, on the other hand, $\kappa = 1$ then we write $\underline{\mathbf{u}} = \underline{\mathbf{u}}' + t\underline{\mathbf{u}}''$ in $\mathscr{N}$, under which transformation $|\Psi_i(\underline{\mathbf{u}})| < q^{d-1}$ is equivalent to $\Psi_i(\underline{\mathbf{u}}'') = 0$, for $i \leqslant n$. Applying the Lang–Weil estimate to this system of equations, we therefore deduce that $\mathscr{N} = O(q^{(1+\kappa)(d-1)n-n})$ for $\kappa \in \{0, 1\}$. An application of (5.7) now completes the proof of the lemma. □

6. The contribution from the minor arcs

We assume that $d \geqslant 3$ throughout this section. Our goal is to prove (3.6) for all $e \geqslant 1$, provided that $n > n_0(d)$, where $n_0(d)$ is given by (1.2). The overall contribution to (3.4) from $|\theta| < q^{-3de}$ is easily seen to be negligible. Hence we may redefine the minor arcs to incorporate the condition $|\theta| \geqslant q^{-3de}$. For $\alpha, \beta \in \mathbb{Z}_{\geqslant 0}$, let $E(\alpha, \beta)$ denote the overall contribution to $N_{\text{minor}}(P)$, from values of a, r, θ for which $|r| = q^\alpha$ and $|\theta| = q^{-\beta}$. The contribution is empty unless

$$0 \leqslant \alpha \leqslant \frac{d(e+1)}{2} \quad \text{and} \quad \alpha + \frac{d(e+1)}{2} \leqslant \beta \leqslant 3de, \tag{6.1}$$

with $\beta \leqslant de+1$ if $\alpha = 0$. Since there are only finitely many choices of α, β, in order to prove (3.6), it will suffice to show that

$$\lim_{q\to\infty} q^{-\widehat{\mu}} E(\alpha, \beta) = 0,$$

for each pair (α, β) under consideration, assuming that $n > n_0(d)$. To begin with, summing trivially over a, we have

$$E(\alpha, \beta) \leqslant q^{2\alpha-\beta+1} \max_{\substack{a,r,\theta \\ |a|<|r|=q^\alpha \\ |\theta|=q^{-\beta}}} |S(a/r+\theta)|. \tag{6.2}$$

We start by dealing with generic values of α and β. Lemma 5.3 implies that

$$E(\alpha, \beta) \leqslant c_{d,n} q^{2\alpha-\beta+1+(e+1)n-L(e+1)\eta},$$

where $L = 2^{-d+1}n$. Recalling the definition (3.2) of $\widehat{\mu}$, the exponent of q is $\widehat{\mu} - \widehat{\nu}$, with

$$\begin{aligned} \widehat{\nu} &= \{(n-d)e + n - 1\} - \{2\alpha - \beta + 1 + (e+1)n - L(e+1)\eta\} \\ &= L(e+1)\eta + \beta - de - 2\alpha - 2. \end{aligned} \tag{6.3}$$

For the choice of η in (5.5), and $n > n_0(d)$, we want to determine when $\widehat{\nu} > 0$. Returning to (5.4), we now see that

$$\Gamma = \frac{1}{d-1} \min\left\{(e+1)(d-1) - 1, \beta - \alpha - 1, (e+1)d - \alpha - 1, \alpha + M\right\},$$

where $M = \max\{0, (e+1)d - \beta\}$. The remainder of the argument is a case by case analysis. When $[\Gamma] \leqslant 1$ we shall return to (6.2), and argue differently based instead on Lemmas 5.4 and 5.5.

Case 1: $\alpha \geqslant 2(d-1)$ and $\beta \geqslant (e+1)d+1$. In this case $M = 0$. Using (6.1), one finds that

$$\Gamma = \frac{1}{d-1} \times \begin{cases} \alpha & \text{if } \alpha < \frac{d(e+1)}{2}, \\ \alpha - 1 & \text{if } \alpha = \frac{d(e+1)}{2}. \end{cases}$$

Let $\iota \in \{0, 1\}$. We write $\alpha - \iota = k(d-1) + \ell$, for $k \in \mathbb{Z}_{\geqslant 0}$ and $\ell \in \{0, \ldots, d-2\}$. Then (5.5) implies that $(e+1)\eta = k - \delta$, where

$$\delta = \begin{cases} 0 & \text{if } k \not\equiv e \bmod 2, \\ 1 & \text{if } k \equiv e \bmod 2. \end{cases} \tag{6.4}$$

We claim that the assumption $\alpha \geqslant 2(d-1)$ implies that $k \geqslant 2$, or else $k = 1$ and $\delta = 0$. This is obvious when $\alpha < \frac{d(e+1)}{2}$. Suppose that $k = 1$ and

$\alpha = \frac{d(e+1)}{2}$. Then $\iota = 0$ and $\ell = d-2$, whence $\alpha = 2(d-1) = \frac{d(e+1)}{2}$. Since $d \geqslant 3$, this equation has no solutions in odd integers e. Thus $\delta = 0$.

Recalling (6.3) and substituting for α, we find that

$$\begin{aligned}\widehat{\nu} &= L(k-\delta) + \beta - de - 2 - 2\iota - 2k(d-1) - 2\ell \\ &= (L - 2(d-1))k - \delta L + \beta - de - 2 - 2\iota - 2\ell \\ &\geqslant (L-2(d-1))k - \delta L - d + 3 - 2\iota,\end{aligned}$$

since $\beta \geqslant (e+1)d + 1$ and $\ell \leqslant d-2$. Taking $3 - 2\iota \geqslant 0$, we have therefore shown that $\widehat{\nu} \geqslant \widehat{\nu}_0$, with

$$\widehat{\nu}_0 = (L - 2(d-1))k - \delta L - d.$$

If $k \geqslant 2$, then we take $\delta \leqslant 1$ to conclude that

$$\widehat{\nu}_0 \geqslant (2-\delta)L - 4(d-1) - d \geqslant L - 5d + 4.$$

Thus $\widehat{\nu}_0 > 0$ if $n > n_0(d)$. Alternatively, if $k = 1$ then we must have $\delta = 0$. It follows that

$$\widehat{\nu}_0 = L - 2(d-1) - d = L - 3d + 2,$$

whence $\widehat{\nu}_0 > 0$ if $n > n_0(d)$, since $n_0(d) \geqslant 2^{d-1} \cdot (3d-2)$ in (1.2).

Case 2: $\alpha + de - d + 2 > \beta$ and $\beta \leqslant (e+1)d$. In this case $M = (e+1)d - \beta$. It follows from (6.1) that

$$\Gamma = \frac{1}{d-1} \times \begin{cases} \alpha + (e+1)d - \beta & \text{if } \beta > 2\alpha, \\ \alpha + (e+1)d - \beta - 1 & \text{if } \beta \leqslant 2\alpha. \end{cases}$$

We proceed as before. Thus for $\iota \in \{0,1\}$, we write

$$\alpha + (e+1)d - \beta - \iota = k(d-1) + \ell, \tag{6.5}$$

with $k \in \mathbb{Z}_{\geqslant 0}$ and $\ell \in \{0, \dots, d-2\}$. Then (5.5) implies that $(e+1)\eta = k - \delta$, where δ is given by (6.4). If $k \geqslant 2$ then (6.3) yields

$$\begin{aligned}\widehat{\nu} &= L(k-\delta) - \beta + de - 2 - 2\iota - 2k(d-1) - 2\ell + 2d \\ &= (L - 2(d-1))k - \delta L - \beta + de - 2 - 2\iota - 2\ell + 2d \\ &\geqslant L - 4d + 4 - \beta + de,\end{aligned}$$

since $\delta, \iota \leqslant 1$ and $\ell \leqslant d-2$. But $\beta \leqslant (e+1)d$, and so it follows that $\widehat{\nu} \geqslant L - 5d + 4$, which is positive if $n > n_0(d)$. Suppose that $k \leqslant 1$. Then, on taking $\iota \leqslant 1$ and $\ell \leqslant d-2$ in (6.5), we must have that

$$\alpha + de - d + 2 \leqslant \beta,$$

which contradicts the hypothesis.

Case 3: $\alpha \leqslant 2(d-1)$ and $\beta \geqslant (e+1)d+1$. In this case we return to (6.2), and we recall the definition (5.6) of κ. Suppose first that $\alpha = 0$. It follows from Lemma 5.5 that $S(a/r+\theta) \ll |P|^n q^{-L}$ if

$$1+\kappa(d-1) \leqslant \beta \leqslant de+1.$$

The upper bound $\beta \leqslant de+1$ follows from the definition of the minor arcs when $\alpha = 0$. Moreover, the lower bound holds, since for $e \geqslant 1$ it follows from (6.1) that $\beta \geqslant d \geqslant 1+\kappa(d-1)$. Recalling (3.2), we conclude that

$$E(\alpha,\beta) \ll q^{-\beta+1+(e+1)n-L} = q^{\widehat{\mu}-\widehat{\nu}},$$

with $\widehat{\nu} = L+\beta-de-2 \geqslant L > 0$, which is satisfactory.

Suppose next that $\alpha \geqslant 1$. Then $S(a/r+\theta) \ll |P|^n q^{-L}$, by Lemma 5.4, provided that

$$e \geqslant 1, \quad 1 \leqslant \alpha < de+1-\kappa(d-1) \quad \text{and} \quad \alpha-\beta < -\kappa(d-1), \tag{6.6}$$

or

$$e = 1, \quad 2 \leqslant \alpha \leqslant d \quad \text{and} \quad \alpha-\beta \leqslant -d. \tag{6.7}$$

In view of (6.1), it is easily seen that $\alpha-\beta < -(d-1) \leqslant -\kappa(d-1)$. Next, we claim that $2d-2 < de+1-\kappa(d-1)$ for any $e \geqslant 2$. This is enough to confirm (6.6), since $\alpha \leqslant 2(d-1)$. The claim is obvious when $\kappa = 1$ and $e \geqslant 3$. On the other hand, if $\kappa = 0$ then $e \geqslant 2$ and it is clear that $2d-2 \leqslant 2d+1 \leqslant de+1$. Next, suppose that $e = 1$, so that $\kappa = 1$. If $\alpha = 1$ then we are plainly in the situation covered by (6.6). If $\alpha \geqslant 2$, on the other hand, then (6.1) implies that $\alpha \leqslant d$ and $\alpha-\beta \leqslant -d$, so that we are in the case covered by (6.7). It follows that

$$E(\alpha,\beta) \ll q^{2\alpha-\beta+1+(e+1)n-L} = q^{\widehat{\mu}-\widehat{\nu}},$$

with

$$\begin{aligned}\widehat{\nu} = L+\beta-de-2-2\alpha &\geqslant L+d-1-2\alpha\\ &\geqslant L-3d+3,\end{aligned}$$

since $\alpha \leqslant 2(d-1)$ and $\beta \geqslant (e+1)d+1$. This is positive for $n > n_0(d)$.

Case 4: $\alpha+de-d+2 \leqslant \beta$ and $\beta \leqslant (e+1)d$. We begin as in the previous case. If $\alpha = 0$, the same argument goes through, leading to $E(\alpha,\beta) \ll q^{\widehat{\mu}-\widehat{\nu}}$, with $\widehat{\nu} = L+\beta-de-2 \geqslant L-d$. This is certainly positive for $n > n_0(d)$. Suppose next that $\alpha \geqslant 1$. Then $S(a/r+\theta) \ll |P|^n q^{-L}$, by Lemma 5.4, provided that (6.6) or (6.7) hold. Note that

$$\alpha \leqslant \beta-de+d-2 \leqslant 2d-2 < de+1-\kappa(d-1),$$

for any $e \geqslant 2$, by the calculation in the previous case. Likewise, the previous argument shows that we are covered by (6.6) or (6.7) when $e = 1$. Thus we find that $E(\alpha,\beta) \ll q^{\widehat{\mu}-\widehat{\nu}}$, with

$$\begin{aligned}\widehat{\nu} = L + \beta - de - 2 - 2\alpha &\geqslant L - d - \alpha \\ &\geqslant L - 3d + 2,\end{aligned}$$

since $\alpha \leqslant 2(d-1)$. This is also positive for $n > n_0(d)$.

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