A FAST ALGORITHM TO COMPUTE $L(1/2, f \times \chi_q)$

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Abstract. Let f be a fixed (holomorphic or Maass) modular cusp form. Let χ_q be a Dirichlet character mod q. We describe a fast algorithm that computes the value $L(1/2, f \times \chi_q)$ up to any specified precision. In the case when q is smooth or highly composite integer, the time complexity of the algorithm is given by $O(1 + |q|^{5/6})$.

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1. Introduction

Let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Let f be a fixed (holomorphic or Maass) cusp form on $\Gamma\backslash\mathbb{H}$. Let χ_q be a Dirichlet character on $\mathbb{Z}/q\mathbb{Z}$. In this paper we consider the problem of computing central values of L-function corresponding f twisted by χ_q. The main result can be summarized as:

Theorem 1. *Let q, M, N be positive integers such that $q = MN$, where $M \leq N$, $M = M_1M_2$ such that $M_1|N$ and $(M_2, N) = 1$. Let f be a modular (holomorphic or Maass) form on $\Gamma\backslash\mathbb{H}$, $s \in \mathbb{H}$ and χ_q a Dirichlet character on $\mathbb{Z}/q\mathbb{Z}$. Let γ, ϵ be any positive reals. Let*

$$E = \min\{M^5 + N, q\}.$$

Then we can compute $L(s, f \times \chi_q)$ up to an error of $O(q^{-\gamma})$ in time $O(E^{1+\epsilon})$. The constants involved in O are polynomial in $(1 + \gamma)/\epsilon$.

This method gives us a positive time saving if q has a factor less than $q^{1/5}$. The maximum saving of size $O(q^{1/6})$ can be obtained if q has a suitable factor of size $\approx q^{1/6}$. In particular, we can get a saving of size $O(q^{1/6})$ for a "smooth" or "highly composite" integer q. Note that for these choices of q, the algorithm is considerably faster than the $O(q^{1+o(1)})$ complexity "approximate functional equation" based algorithms.

1.1. **Model of computation.** We will use the real number (infinite precision) model of computation that uses real numbers with error free arithmetic having cost as unit cost per operation. An operation here means addition, subtraction, division, multiplication, evaluation of logarithm (of a complex number z such that $|\arg(z)| < \pi$) and exponential of a complex number.

Our algorithm will work if we work with numbers specified by $O(\log q)$ bits. This will at most add a power of $\log q$ in the time complexity of the algorithm. We refer the readers to [20, Chapter 8] and [21] for more details about the real number model of computation.

1.2. **Historical background and applications.** The problem of "computing" values of the zeta function effectively goes as far back as Riemann. Riemann used the Riemann Siegel formula to compute values of the zeta function and verify the Riemann hypothesis for first few zeroes. The Riemann Siegel formula writes $\zeta(1/2 + iT)$ as a main sum of length $O(T^{1/2})$ plus a small easily "computable" error. Subsequent improvements for the rapid evaluation of zeta are considered in Schönhage and Odlyzko [13], Schönhage [18], Hiary [9] and [8], S Turing [22], Berry and Keating [2], Rubinstein [15] and Arias De Reyna [1] *et al.* The current fastest algorithm for evaluating $\zeta(1/2 + iT)$ for a single value of T is due to Hiary (time complexity $O(T^{4/13+o(1)})$, see [8]). The main idea behind these algorithms is to start with the main sum in the Riemann Siegel formula, cut the sum into "large" number of subsums having "large enough" lengths and try to compute these subsums rapidly.

The next natural problem to consider is computing $L(1/2 + iT, \chi_q)$, where χ_q is a Dirichlet character modulo an integer q. A $O(T^{1/3+o(1)}q^{1/3+o(1)})$ algorithm for highly composite q, is given by Hiary in [10]. In this algorithm, the rapid computation of $L(1/2, \chi_q)$ is essentially reduced to the problem of fast computations of character sums $\sum_{k=k_0}^{k_0+M} \chi_q(k)$, for any k_0 and for a M, which is a small power of q. In case of highly composite q, one exploits highly repetitive nature of χ_q to get a fast way of computing these character sums. The problem of computing $L(1/2, \chi_q)$ for an "almost prime" q seems to be rather difficult.

In the case of higher rank L-functions, the analogue of the Riemann Siegel formula is given by the approximate functional equation. A detailed description of "the approximate functional equation" based algorithms is given by Rubinstein in [15]. In the case of L-function associated to a modular (holomorphic or Maass) form, these algorithms have $O(T^{1+o(1)})$ time complexity. The algorithms for rapid computation of the GL(1) L- functions unfortunately do not readily generalize to the higher rank cases, due to the complicated main sum in the usual approximate functional equation. A geometric way for computing $L(f, 1/2 + iT)$ in time $O(T^{7/8+o(1)})$, for a modular form f is given in [25].

In this paper we consider the higher rank problem corresponding to [10], *i.e* computing $L(1/2, f \times \chi_q)$. We give a $O(q^{5/6+o(1)})$ complexity algorithm for a "smooth" or "highly composite" q. Our algorithm in theorem 1 is the first known improvement of the approximate functional equation based algorithms in in $\mathrm{GL}(2) \times \mathrm{GL}(1)$ setting.

Computing values of L-functions on the critical line has various applications in number theory. It can be used to verify the *Generalized Riemann Hypothesis* numerically. It has also been used to connect the distribution of values of L-functions on the critical line to the distributions of eigenvalues of unitary random matrices via the recent random matrix theory conjectures.

The problem of computing L-functions is closely related to the problem of finding subconvexity bounds for the L-functions. More generally, improving the "square root of analytic conductor bounds" coming from the approximate functional equation is of great interest to analytic number theorists.

In the present paper, we have only considered the $\mathrm{GL}_2 \times \mathrm{GL}_1$ case. It will be of great interest to generalize the method in this paper to higher rank L-functions. Integral representations for a more general class of $\mathrm{GL}_2 \times \mathrm{GL}_1$ L-functions is given in [23, section 11.4]. Our method thus could also generalize for L-functions $L(1/2, f \times \chi)$, in the number field setting. A more interesting problem will be to generalize our technique in the $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ setting, where the subconvexity bounds are also not known.

1.3. **Outline of the proof.** Our algorithm starts with writing $L(1/2, f \times \chi_q)$ essentially as a sum

$$\sum_{j=0}^{q-1} f(j/q + i/q)\chi_q(j),$$

on the q^{th} Hecke orbit of i. The results in this paper are closely related to the work of Venkatesh [23, section 6], where he used the equidistribution of the points $\{j/q + i/q : 0 \le j \le q-1\}$ in $\Gamma\backslash\mathbb{H}$, to get subconvexity bounds for the L-functions. We however use the fact that for a composite $q = MN$, we write

$$\{j/q + i/q : 0 \le j \le q-1\} = \cup_{j=0}^{N-1}\{(j + kN)/q + i/q : 0 \le k \le M-1\}.$$

Each arithmetic progression $\{(j + kN)/q + i/q : 0 \le k \le M-1\}$ can be viewed as part of the M^{th} Hecke orbit of the point $j/N + i/N$.

We then "sort" the points $\{j/N + i/N\}$ into sets $K_1, K_2, ..$ such that the points in each set are very close to one other in $\Gamma\backslash\mathrm{SL}(2, \mathbb{R})$. We use the fact that Hecke orbits of close enough points in the upper half plane remain close, along with "well behaved nature" of χ_q on these arithmetic progressions to get a fast way of numerically computing $L(1/2, f \times \chi_q)$, up to any given precision.

1.4. **Outline of the paper.** A brief account of the notations used in this paper is given in section 2. The algorithm uses a type of "geometric approximate functional equation". It is discussed in detail in section 3. A detailed proof of theorem 1 is given in section 4. Lemmas 4.1 and 2.5 deal with well behaved nature of Hecke orbits of nearby points in $\mathbb{H}$ and well behaved nature of χ_q on the "arithmetic progressions" respectively. They are proved in section 5.

2. Notation and preliminaries

Throughout, let $\Gamma = \mathrm{SL}(2, \mathbb{Z})$. Let f be a holomorphic or Maass cusp form on $\Gamma\backslash\mathbb{H}$.

Let s be a fixed point in $\mathbb{H}$. Throughout, let q be a positive integer. Let χ_q be a Dirichlet character on $\mathbb{Z}/q\mathbb{Z}$. Let γ, ϵ be any given positive numbers, independent of q. In practice, γ will be taken to be $O(1)$ and ϵ will be a small positive number.

Given a Dirichlet character χ_q, The Gauss sum $\tau(\chi_q)$ is defined by

$$\tau(\chi_q) = \sum_{k=0}^{q-1} \chi_q(k) e(k/q). \tag{2.1}$$

We will denote the set of nonnegative integers by $\mathbb{Z}_+$ and the set of nonnegative real numbers by $\mathbb{R}_+$.

We will use the symbol $\ll$ as is standard in analytic number theory: namely, $A \ll B$ means that there exists a positive constant c such that $A \leq cB$. These constants will always be independent of the choice of T.

We will use the following special matrices in $\mathrm{SL}(2, \mathbb{R})$ throughout the paper:

$$n(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, a(y) = \begin{pmatrix} e^{y/2} & 0 \\ 0 & e^{-y/2} \end{pmatrix}, K(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.2}$$

$e(x)$ will be used to denote $\exp(2\pi i x)$.

In this paper, for simplicity let us assume that f is either holomorphic or an even Maass form. The algorithms will be analogous for the odd Maass cusp form case. For an even Maass cusp form, we will use the following power series expansion :

$$f(z) = \sum_{n>0} \hat{f}(n) W_r(nz). \tag{2.3}$$

Here $W_r(x+iy) = 2\sqrt{y} K_{ir}(2\pi y) \cos(2\pi x)$. The explicit Fourier expansion for the holomorphic cusp forms is given by

$$f(z) = \sum_{n>0} \hat{f}(n) e(nz). \tag{2.4}$$

For a cusp form of weight k (for the case of Maass forms k is assumed to be 0), the corresponding twisted L- function is defined by:

Definition 2.1. $L(s, f \times \chi_q) = \sum_{n=1}^{\infty} \frac{\hat{f}(n)\chi_q(n)}{n^{s+(k-1)/2}}$.

Given a cusp (Maass or holomorphic) form of weight k on $\Gamma\backslash\mathbb{H}$, we will define a lift $\tilde{f}$ of f to $\Gamma\backslash\mathrm{SL}(2, \mathbb{R})$ by $\tilde{f} : \Gamma\backslash\mathrm{SL}(2, \mathbb{R}) \to \mathbb{C}$ such that

$$\tilde{f}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (ci + d)^{-k} f(\frac{ai + b}{ci + d}). \tag{2.5}$$

2.1. **Real analytic functions on $\Gamma\backslash\mathrm{SL}(2, \mathbb{R})$.** We will use the same notation as [25] for real analytic functions on $\Gamma\backslash\mathrm{SL}(2, \mathbb{R})$.

Let x be an element of $\mathrm{SL}(2, \mathbb{R})$ and let g be a function on $\Gamma\backslash\mathrm{SL}(2, \mathbb{R})$, *a priori* $g(x)$ does not make sense but throughout we abuse the notation to define

$$g(x) = g(\Gamma x).$$

i.e. $g(x)$ simply denotes the value of g at the coset corresponding to x.

Let ϕ be the Iwasawa decomposition given by

$$\phi : (t, y, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to n(t)a(y)K(\theta).$$

Recall that ϕ restricted to the set $\mathbb{R} \times \mathbb{R} \times (-\pi, \pi]$ gives a bijection with $\mathrm{SL}(2, \mathbb{R})$.

Definition 2.2. *Given $\eta > 0$, let $\mathfrak{U}_\eta = (-\eta, \eta) \times (-\eta, \eta) \times (-\eta, \eta)$ and $U_\eta = \phi(\mathfrak{U}_\eta) \subset \mathrm{SL}(2, \mathbb{R})$.*

Let us define the following notion of "derivatives" for smooth functions on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$:

Definition 2.3. *Let g be a function on $\mathrm{SL}(2,\mathbb{R})$ and x any point in $\mathrm{SL}(2,\mathbb{R})$. We define (wherever R.H.S. makes sense)*

$$\frac{\partial}{\partial x_1} g(x) = \frac{\partial}{\partial t}|_{t=0} g(xn(t));$$
$$\frac{\partial}{\partial x_2} g(x) = \frac{\partial}{\partial t}|_{t=0} g(xa(t));$$
$$\frac{\partial}{\partial x_3} g(x) = \frac{\partial}{\partial t}|_{t=0} g(xK(t)).$$

Sometimes, we will also use ∂_i to denote $\frac{\partial}{\partial x_i}$.

Given $\beta = (\beta_1, \beta_2, \beta_3)$, let us define $\partial^\beta g(x)$ by

$$\partial^\beta g(x) = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} \frac{\partial^{\beta_3}}{\partial x_3^{\beta_3}} g(x). \tag{2.6}$$

For β as above we will define

$$\beta! = \beta_1!\beta_2!\beta_3!$$

and

$$|\beta| = |\beta_1| + |\beta_2| + |\beta_3|.$$

We now define the notion of real analyticity as follows:

A function g on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ will be called real analytic, if given any point x in $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, there exists a positive real number r_x such that g has a power series expansion given by

$$g(xn(t)a(y)K(\theta)) = \sum_{\beta=(\beta_1,\beta_2,\beta_3)\in\mathbb{Z}_+^3} \frac{\partial^\beta g(x)}{\beta!} t^{\beta_1} y^{\beta_2} \theta^{\beta_3} \tag{2.7}$$

for every $(t, y, \theta) \in \mathfrak{U}_{r_x}$.

Let us use the following notation for the power series expansion.

Definition 2.4. *Let $y, x \in \mathrm{SL}(2,\mathbb{R})$ and t, y, θ be such that $y = xn(t)a(y)K(\theta)$ and $(\beta_1, \beta_2, \beta_3) = \beta \in \mathbb{Z}_+^3$ define*

$$(y - x)^\beta = t^{\beta_1} y^{\beta_2} \theta^{\beta_3}.$$

Hence we can rewrite the Equation (2.7) as

$$g(y) = \sum_{\beta=(\beta_1,\beta_2,\beta_3),\beta\in\mathbb{Z}_+^3} \frac{\partial^\beta g(x)}{\beta!} (y - x)^\beta.$$

Throughout, we will assume that for a cusp form f, all the derivatives of the lift $\tilde{f}$ (of f) are bounded uniformly on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$ by 1. In general it can be proved that given a cusp form f, there exists R such that $||\partial^\beta \tilde{f}||_\infty \ll R^{|\beta|}$, see [24, section 8.2]. The case when $R > 1$ can be dealt with analogously. The assumption that all derivatives are bounded by 1, allows the proofs to be marginally simpler.

2.2. **Hecke orbits.** Let L be any positive integer and $x \in \mathrm{SL}(2,\mathbb{R})$, let

$$T(L) = \{(m,k) : m|L, 0 \leq k < L/m\}$$

and

$$A(L,m,k) = \frac{1}{L^{1/2}} \begin{pmatrix} m & k \\ 0 & L/m \end{pmatrix}.$$

The L^{th} Hecke orbit is given by left action of the cosets $\{\Gamma A(L,m,k) : (m,k) \in T(L)\}$. In particular, the L^{th} Hecke orbit of x is given by $\{\Gamma A(L,m,k)x, (m,k) \in T(L)\}$, considered as a subset of $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. The Hecke orbits generalize the notion of an "arithmetic progression" on $\mathrm{SL}(2,\mathbb{R})$.

It is well known that the right action of any element a in $\mathrm{SL}(2,\mathbb{Z})$ permutes the cosets $\{\Gamma A(L,m,k) : (m,k) \in T(L)\}$. This implies that

$$\{\Gamma A(L,m,k) : (m,k) \in T(L)\} = \{\Gamma A(L,m,k)a : (m,k) \in T(L)\}$$

for any $a \in \mathrm{SL}(2,\mathbb{Z})$. Let $\sigma_{\mathfrak{a}} : T(L) \to T(L)$ be be the permutation defined by

$$\Gamma A(L,m,k)a = \Gamma A(L, \sigma_{\mathfrak{a}}(m,k)).$$

Given any $\epsilon > 0$, using the fact that the number of divisors of M is at most $O(M^{\epsilon})$, we get that the cardinality of $T(M)$ is at most $O(M^{1+\epsilon})$. *A priori* there are $M^{1+\epsilon}!$ possible permutations on $T(L)$. However it is easy to prove the following lemma (see section 5):

Lemma 2.5. *Let a_1 and a_2 are matrices in $\mathrm{SL}(2,\mathbb{Z})$ such that $a_1 \equiv a_2 \bmod L$. Then the corresponding permutations σ_{a_1} and σ_{a_2} are equal. In other words, for every $(m,k) \in T(L)$, $\sigma_{a_1}(m,k) = \sigma_{a_2}(m,k)$.*

Lemma 2.5 implies that the number of possible permutations of the Hecke orbit $\{\Gamma A(L,m,k)|(m,k) \in T(L)\}$ due to the right action of $\mathrm{SL}(2,\mathbb{Z})$ are at most $|\mathrm{SL}(2,\mathbb{Z}/L\mathbb{Z})| \leq L^3$.

2.3. **Specifying f and χ_q.** We will follow the same assumptions for input of f as in [25]. In particular, we will assume that each value of f (or $\tilde{f}$) or any of it's derivative can be computed exactly in time $O(1)$. [1]

We will also assume that given any integer n, $\chi_q(n)$ can be computed in time $O(1)$. It can be easily be shown that for $q = MN$, using and storing a precomputation of size $O(M+N)$, one can compute any $\chi_q(n)$, in further $O(\log q)$ steps. The time complexity $O(M+N)$ in precomputation does not change the asymptotics of the algorithm. Similarly, allowing $\log(q)$ time for each valuation of χ_q only adds a multiple of $\log q$ to the time complexity of the algorithm, which can be absorbed into the exponent ϵ.

In practice, the Gauss sum $\tau(\chi_q)$ can be computed rather rapidly. A very simple $O(M^2+N)$ time complexity algorithm can be found in [24, section 8.6]. As in the previous case, this does not change the asymptotic time complexity of the algorithm. It is worth mentioning to the reader that the algorithm in [24, section 8.6] is similar to the algorithm in the paper and can be helpful in better understanding of the underlying idea behind it.

[1] It can be easily shown that given any x and any fixed γ, one can compute $\partial^{\beta}\tilde{f}(x)$ up to the error $O(q^{-\gamma})$ in $O(q^{o(1)})$ time. Here the constant involved in O is a polynomial in $|\beta|$ and γ. In this algorithm we only compute values of $\partial^{\beta}\tilde{f}(x)$ for $|\beta| \ll 1$. Allowing $O(q^{o(1)})$ time for each valuation of $\tilde{f}$ does not change the time complexity of the algorithm. See [24, chapter 7] for explicit details about it

3. A geometric approximate functional equation for $L(s, f \times \chi_q)$

Our algorithm will start with proving a "geometric approximate functional equation" for $L(s, f \times \chi_q)$, given by (3.12) and (3.13). The right hand side of (3.12) and (3.13) consists of sums of q integrals. Each of these integrals is an integral of a 'nice' function on a geodesic of (hyperbolic) length $O(\log q)$. Therefore, using [25, proposition 8.1], we can write each of these integrals (up to an error of $O(q^{-\gamma})$), as a sum of size $O(q^{\epsilon})$ terms. Adding all these sums together, the right hand sides of (3.12) and (3.13) can be written (up to an error of $O(q^{-\gamma})$) a sum of size $O(q^{1+\epsilon})$. The constants involved in O are polynomial in $(1+\gamma)/\epsilon$ and are independent of q.

Holomorphic case:

$$\begin{aligned}\sum_{k=0}^{q-1} \chi_q(k) f(k/q + iy) &= \sum_{k=0}^{q-1} \chi_q(k) \sum_{n=1}^{\infty} \hat{f}(n) e(nk/q) e(iny) \\ &= \sum_{n=1}^{\infty} \hat{f}(n) e(iny) \sum_{k=0}^{q-1} \chi_q(k) e(nk/q) \\ &= \tau(\chi_q) \sum_{n=1}^{\infty} \hat{f}(n) \chi_q(n) e(iny). \end{aligned} \tag{3.1}$$

After taking the Mellin transform of (3.1), we get

$$\begin{aligned}&\sum_{k=0}^{q-1} \chi_q(k) \int_0^{\infty} f(k/q + iy) y^{s+(k-3)/2} dy \\ &= \tau(\chi_q) \sum_{n=1}^{\infty} \hat{f}(n) \chi_q(n) \int_0^{\infty} e(iny) y^{s+(k-3)/2} dy; \\ &= \tau(\chi_q) \sum_{n=1}^{\infty} \hat{f}(n) \chi_q(n) \int_0^{\infty} \exp(-2\pi ny) y^{s+(k-3)/2} dy; \\ &= \frac{\tau(\chi_q)}{(2\pi)^{s+(k-1)/2}} L(s, f \times \chi_q) \Gamma(s + (k-1)/2). \end{aligned} \tag{3.2}$$

Even Maass form case:

$$\begin{aligned}&\sum_{k=0}^{q-1} \chi_q(k) f(k/q + iy) \\ &= 2 \sum_{k=0}^{q-1} \chi_q(k) \sum_{n=1}^{\infty} \hat{f}(n) \sqrt{ny} \cos(2\pi nk/q) K_{ir}(2\pi ny) \\ &= 2 \sum_{n=1}^{\infty} \hat{f}(n) \sqrt{ny} K_{ir}(2\pi ny) \sum_{k=0}^{q-1} \chi_q(k) \frac{e(nk/q) + e(-nk/q)}{2} \\ &= 2 \frac{\tau(\chi_q) + \overline{\tau(\overline{\chi_q})}}{2} \sum_{n=1}^{\infty} \hat{f}(n) \chi_q(n) \sqrt{ny} K_{ir}(2\pi ny) \\ &= \tau(\chi_q)(1 + \chi_q(-1)) \sum_{n=1}^{\infty} \hat{f}(n) \chi_q(n) \sqrt{ny} K_{ir}(2\pi ny). \end{aligned} \tag{3.3}$$

Taking Mellin transform of (3.3), we get

$$\sum_{k=0}^{q-1}\chi(k)\int_0^\infty f(k/q+iy)y^{s-3/2}dy \tag{3.4}$$
$$= 2\tau(\chi_q)\frac{1+\chi_q(-1)}{2}\sum_{n=1}^{\infty}\hat{f}(n)\sqrt{n}\chi_q(n)\int_0^\infty K_{ir}(2\pi ny)y^{s-1}dy;$$
$$= \tau(\chi_q)\frac{1+\chi_q(-1)}{4(\pi)^s}L(s,f\times\chi_q)\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right).$$

$\tau(\chi_q)$ is a complex number with absolute value $q^{\frac{1}{2}}$. This implies that we can use (3.2) to compute $L(s,f\times\chi_q)$ in the holomorphic case.

For an even Maass form f however, if $\chi_q(-1)=-1$, then the right hand side of (3.4) is zero. Therefore it needs slightly different treatment. Recall the Fourier expansion for f given by

$$f(z)=\sum_{n>0}\hat{f}(n)2\sqrt{y}K_{ir}(2\pi ny)\cos(2\pi nx).$$

This implies that

$$\partial_x f(z)=-2\pi\sum_{n>0}n\hat{f}(n)2\sqrt{y}K_{ir}(2\pi ny)\sin(2\pi nx).$$

We use a similar method as before to get:

$$\sum_{k=0}^{q-1}\chi_q(k)\partial_x f(k/q+iy)$$
$$= -2\pi\sum_{k=0}^{q-1}\chi_q(k)\sum_{n=1}^{\infty}n\sqrt{ny}\hat{f}(n)\sin(2\pi nk/q)K_{ir}(2\pi ny)$$
$$= -2\pi\sum_{n=1}^{\infty}\hat{f}(n)n^{3/2}\sqrt{y}K_{ir}(2\pi ny)\sum_{k=0}^{q-1}\chi_q(k)\frac{e(nk/q)-e(-nk/q)}{2i}$$
$$= i\pi(\tau(\chi_q)-\overline{\tau(\overline{\chi_q})})\sum_{n=1}^{\infty}\hat{f}(n)n^{3/2}\chi_q(n)\sqrt{y}K_{ir}(2\pi ny)$$
$$= i\pi\tau(\chi_q)(1-\chi_q(-1))\sum_{n=1}^{\infty}\hat{f}(n)n^{3/2}\chi_q(n)\sqrt{y}K_{ir}(2\pi ny). \tag{3.5}$$

Taking Mellin transform of (3.5), we get

$$\sum_{k=0}^{q-1}\chi(k)\int_0^\infty \partial_x f(k/q+iy)y^{s-1/2}dy \tag{3.6}$$
$$= i\pi\tau(\chi_q)(1-\chi_q(-1))\sum_{n=1}^{\infty}\hat{f}(n)\chi_q(n)n^{3/2}\int_0^\infty K_{ir}(2\pi ny)y^{s}dy$$
$$= i\pi\tau(\chi_q)\frac{1-\chi_q(-1)}{4(\pi)^{s+1}}L(s,f\times\chi_q)\Gamma\left(\frac{s+1+ir}{2}\right)\Gamma\left(\frac{s+1-ir}{2}\right).$$

Notice that (3.6) is analogous to (3.4). The algorithm to compute $\sum_{k=0}^{q-1} \chi(k) \int_0^\infty \partial_x f(k/q+iy) y^{s-1/2} dy$ is completely analogous to the algorithm to compute $\sum_{k=0}^{q-1} \chi(k) \int_0^\infty f(k/q+iy) y^{s-3/2} dy$. Hence throughout the rest of the paper, we will assume that $\chi_q(1) = \chi_q(-1) = 1$. Using the automorphy of f, we get the following lemma (analogous to [25, Lemma 3.1]).

Note that a similar treatment will give us the corresponding "geometric approximate functional equations" for odd Maass forms.

Lemma 3.1. *Given any cusp form f (of weight k) on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, positive coprime integers n, q such that $n < q$, χ_q a character* mod$q$*, $s \in \mathbb{H}$, a positive real γ and for any $c > 2$,*

$$\int_0^\infty f(n/q+iy) y^s dy = \int_{q^{-c}}^{q^c} f(n/q+iy) y^s dy + O(q^{-\gamma}). \tag{3.7}$$

The constant involved in O is independent of q.

Proof. Let us use the exponential decay of f at $i\infty$, to get for $y \geq 1$,

$$|f(n/q+iy)| \ll \exp(-\pi y). \tag{3.8}$$

This implies that we can choose a constant c_1, independent of n, q such that for every $y \geq c_1 \log q$, we have

$$|f(n/q+yi)| < q^{-|s|-\gamma-2} y^{-2}. \tag{3.9}$$

Let n', n'' such that $0 \leq n' < q$ and $nn' - qn'' = 1$. The the action of $g = \begin{pmatrix} n' & -n'' \\ -q & n \end{pmatrix}$ on $\mathbb{H}$ maps n/q to infinity. Using the automorphy of f with respect to the action of g, we get

$$\begin{aligned} f(n/q+iy) &= (-q(n/q+iy)+n))^{-k} f\left(\frac{n'(n/q+iy)-n'')}{-q(n/q+iy)+n}\right) \\ &= (-qiy)^{-k} f\left(\frac{1/q+in'y}{-qyi}\right) \\ &= (-qiy)^{-k} f\left(-n'/q+\frac{i}{q^2 y}\right). \end{aligned} \tag{3.10}$$

We use the exponential decay of f at infinity to get a constant c_2, independent of n, q such that for every $y \leq \frac{1}{c_2 q^2 \log q}$, we have

$$|y^{-k} f(n/q+iy)| < q^{-\gamma-|s|-2}. \tag{3.11}$$

The equations (3.9) and (3.11) give us the result.

∎

Lemma 3.1 implies that for any $c > 2$,

$$\begin{aligned} &\frac{\tau(\chi_q)}{(2\pi)^{s+(k-1)/2}} L(s, f \times \chi_q) \Gamma(s+(k-1)/2) \\ &= \sum_{j=0}^{q-1} \chi(j) \int_{q^{-c}}^{q^c} f(j/q+iy) y^{s+(k-3)/2} dy + O(q^{-\gamma}). \end{aligned} \tag{3.12}$$

and similarly we get that given any $q, \gamma > 0$, a Maass cusp form f, and $c > 2$, we have

$$\tau(\chi_q)\frac{1+\chi_q(-1)}{4(\pi)^s}L(s,f\times\chi_q)\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) = \sum_{k=0}^{q-1}\chi(k)\int_{q^{-c}}^{q^c} f(k/q+iy)y^{s-1}dy + O(q^{-\gamma}). \tag{3.13}$$

The equations (3.12) and (3.13) denote "geometric approximate functional equations" to compute $L(s, f\times\chi_q)$ in the holomorphic and Maass form case respectively. Notice that the integrals on the right hand side of (3.12) and (3.13) are over hyperbolic curves of length $\ll \log q$, therefore they can be computed in $O(q^{o(1)})$ time. In the following theorem, we discretize the integrals in these equations to convert the problem of computing the sum on the right hand side of (3.12)/(3.13) to the problem of computing the sum $S = \sum_{j=0}^{q-1}\chi(j)\partial_2^l\tilde{f}(n(j/q)a(t))$ for any $t\in[-3C\log q, 3C\log q]$.

Lemma 3.2. *Let f be a modular (holomorphic or Maass) cusp form on $\Gamma\backslash\mathbb{H}$, and χ_q be a Dirichlet character modulo q, s be any complex number. Let γ,ϵ be any positive reals, then there exists a positive integer $N' = O((1+\gamma)/\epsilon)$ and a constant C such that if for any $|t|\ll\log q$ and for any $0\le l\le N'$, we can compute $\sum_{j=0}^{q-1}\chi(j)\partial_2^l\tilde{f}(n(j/q)a(t))$ up to a maximum error $O(q^{-\gamma})$ in time $D(q)$, then we can compute $\tau(\chi_q)L(s,f\times\chi_q)$ using $O(D(q)q^\epsilon)$ operations. The constants in O are polynomial in $(1+\gamma)/\epsilon$.*

Proof. Let us start with the "geometric approximate functional equations" (3.12)/ (3.13) for holomorphic/Maass case respectively. Notice that the constant c in (3.12)/ (3.13) can be chosen to be greater than 1. We use the lift $\tilde{f}$ defined in (2.5) to get $\tilde{f}(n(x)a(\log y)) = y^{k/2}f(x+iy)$ to rewite (3.12) as:

$$\frac{\tau(\chi_q)}{(2\pi)^{s+(k-1)/2}}L(s,f\times\chi_q)\Gamma(s+(k-1)/2) = \sum_{j=0}^{q-1}\chi(j)\int_{q^{-c}}^{q^c}\tilde{f}(n(j/q)a(\log y))y^{s-3/2}dy + O(q^{-\gamma}).$$

Notice that the above equation will be valid for any $c>2$. Substitute $\log y = t$ in the above equation to get that

$$\frac{\tau(\chi_q)}{(2\pi)^{s+(k-1)/2}}L(s,f\times\chi_q)\Gamma(s+(k-1)/2) = \sum_{j=0}^{q-1}\chi(j)\int_{-c\log q}^{c\log q}\tilde{f}(n(j/q)a(t))e^{t(s-1/2)}dt + O(q^{-\gamma}).$$

Therefore we have

$$\frac{\tau(\chi_q)L(s,f\times\chi_q)\Gamma(s+(k-1)/2)}{(2\pi)^{s+(k-1)/2}} = C'\sum_{j=0}^{q-1}\chi_q(j)\int_{-c\log q}^{c\log q}g_j(t)dt + O(q^{-\gamma}). \tag{3.14}$$

Here $c' = q^{c|\mathrm{Re}(s-\frac{1}{2})|} = \sup_{-c\log q\le t\le c\log q}|\exp(t(s-1/2))|$

$$g_j(t) = \frac{1}{C'}\tilde{f}(n(j/q)a(t))\exp(t(s-1/2)).$$

Notice that $\frac{d}{dt}\mid_{t=t_0} (\tilde{f}(n(j/q)a(t))) = \frac{\partial}{\partial x_2}\tilde{f}(n(j/q)a(t_0))$. Here $\frac{\partial}{\partial x_2}$ denote the derivative of $\tilde{f}$ in the "geodesic direction", defined in section 2.

We have assumed that $\tilde{f}$ has bounded derivatives (ref. section 2). Therefore using Leibniz rule, for a fixed $s \in \mathbb{C}$ and each t in $[-c\log q, \log q]$,

$$\frac{\partial^n}{\partial t^n} g_j(t_0) \ll_f (|s-1/2|+1)^n.$$

The constant involved is independent of q. Hence for each t in $[-3\log q, 3\log q]$, g_j is real analytic with radius of convergence at least $1/(|s-1/2|+1)$. Therefore, choosing a grid of $O(q^\epsilon)$ equispaced points and using power series expansion at the nearest grid point on the left to compute g_k at any given point, we get that given any $\gamma, \epsilon > 0$,

$$\int_{-c\log q}^{c\log q} \tilde{f}(n(j/q)a(t))\exp(t(s-1/2))dt$$

$$= C' \sum_{x=-cq^\epsilon \log q}^{cq^\epsilon(\log q)-1} \sum_{l=0}^{N'} \int_0^{q^{-\epsilon}} \partial_2^l(g_j(xq^{-\epsilon}))\frac{t^l}{l!}dt + O(q^{-1-\gamma}) \tag{3.15}$$

Here $N' = O((1+\gamma)/\epsilon)$. Notice that the above equation is true for any $c > 2$. Hence , given ϵ, we can choose c such that $\{cq^\epsilon \log q\} = 0$. Multiplying (3.15) by $\chi_q(k)$ and summing over k, we get

$$L(s, f\times\chi_q) = C'' \sum_{x=-3Cq^\epsilon\log q}^{3Cq^\epsilon \log q -1} \sum_{l=0}^{N'} d_l \sum_{j=0}^{q-1} \chi_q(j)\partial^l(g_j(xq^{-\epsilon})) + O(q^{-\gamma}) \tag{3.16}$$

Here $d_l = \int_0^{q^{-\epsilon}} \frac{t^l}{l!}dt = \frac{q^{-\epsilon(l+1)}}{(l+1)!}$ and $C'' = \frac{C'(2\pi)^{s+(k-1)/2}}{\Gamma(s+(k-1)/2)\tau(\chi_q)}$.

(3.16), along with the definition of $g_j(t)$ implies that the problem of computing $L(s, f\times\chi)$ is equivalent to computing $\tau(\chi_q)$ and $\sum_{j=0}^{q-1}\chi(j)\partial_2^l\tilde{f}(n(j/q)a(t))$ for any $t \in [-3\log q, 3\log q]$ and for $0 \le l \le N'$ faster than $O(q)$. Here ∂_2 denotes $\frac{\partial}{\partial x_2}$. Exactly same method will work for the even Maass forms, when $\chi_q(-1) = -1$.

Note that for an even Maass form if $\chi_q(-1) = -1$, then we need to compute the sum $\sum_{j=0}^{q-1}\chi(j)\int_{q^{-c}}^{q^c} \partial_x f(j/q+iy)y^{s-1/2}dy$. Notice that $\tilde{f}(n(x)a(\log y)) = f(x+iy)$. Therefore, for any x_0, an easy computation gives

$$\partial_x f(x_0+iy) = y^{-1}\partial_1\tilde{f}(n(x_0)a(\log y)).$$

Using this equation, we can follow exactly the same method before, to get lemma when $\chi_q(-1) = -1$. □

In the following sections, we prove that $D(q) = (M^{5+\epsilon}+N)^{1+\epsilon}$. We use this result to finish the proof of theorem 1

4. Proof of theorem 1

Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$ and f be a (holomorphic or Maass) cusp form of weight k on $\Gamma\backslash\mathbb{H}$. Let $q = MN$ where $M \le N$, $M = M_1M_2$, where $M_1 = \gcd(M,N)$ and $(M_2, N) = 1$. Let χ_q be a Dirichlet character on $\mathbb{Z}/q\mathbb{Z}$. Let γ, ϵ be any given positive numbers. The main goal of this section is to prove theorem 1.

Theorem 3.2 will prove that the algorithm to compute $L(s, f \times \chi_q)$ faster than $O(q)$ is equivalent to computing the Gauss sum $\tau(\chi_q)$ and

$$\sum_{k=0}^{q-1} \chi_q(k)\partial_2^l \tilde{f}(n(k/q)a(t)) \tag{4.1}$$

for any $t \in [-c\log q, c\log q]$ and for $0 \le l \le N'$, faster than $O(q)$. Here ∂_2 denotes $\frac{\partial}{\partial x_2}$, $N' = O((1+\gamma)/\epsilon)$ and $c > 2$ is a suitable computable constant. Hence, in this section we will consider the problem of computing the sum in (4.1) for any given $t \in [-c\log q, c\log q]$.[2]

Computing $\tau(\chi_q)$ can be done very rapidly. It is easy to see that given any positive γ, we can compute $\tau(\chi_q)$ up to error $O(q^{-\gamma})$, using at most $O(M^2 + N)$ operations. The algorithm is given in [24, section 8.6]. It is similar to the algorithm presented in this paper. This algorithm is simpler, and can be helpful in understanding the main algorithm in this paper better.

We will deal with computing (4.1) for the special cases where $q = MN$ where $(M, N) = 1$ and the case when $M|N$ separately in the sections 4.1 and 4.4 respectively. We use these results to complete the proof of theorem 1 in section 4.5.

.

4.1. **Case $q = MN$ where $(M, N) = 1$.** Let g be a real analytic function with bounded derivatives on $\Gamma\backslash\mathrm{SL}(2,\mathbb{R})$. Let $q = MN$ where $(M, N) = 1$. Let t be a real number in $[-c\log q, c\log q]$, $t_1 = t + \log q$. Let

$$x_0 = a(t_1).$$

In this section, we will consider the problem of computing the sum

$$\begin{aligned} S &= \sum_{k=0}^{q-1} \chi_q(k) g(n(k/q)a(t)) \\ &= \sum_{k=0}^{q-1} \chi_q(k) g(A(q,1,k)a(t_1)) \\ &= \sum_{j=0}^{N-1}\sum_{k=0}^{M-1} \chi_q(j+Nk) g(A(q,1,j+kN)x_0). \end{aligned} \tag{4.2}$$

Computing the sum of this type is equivalent to computing the sum (4.1). Recall that $A(q, 1, k)$ is defined in section 2.2. Use, $A(q, 1, j + kN) = A(M, 1, k)A(N, 1, j)$ to rewrite (4.2) as

$$S = \sum_{j=0}^{N-1} S_j. \tag{4.3}$$

[2]Note that for a Maass form f, for $g = \tilde{f}$, $t = -\log q$ and $l = 0$, (4.1) takes a more familiar form

$$\sum_{k=0}^{q-1} \chi_q(k) f(k/q + i/q).$$

In this section, we essentially give an algorithm to compute $\sum_{k=0}^{q-1} \chi_q(k) f(k/q + i/q)$, faster than $O(q)$.

Here, S_j is defined as

$$S_j = \sum_{k=0}^{M-1} \chi_q(j+kN) g(A(M,1,k)v_j). \tag{4.4}$$

Here

$$v_j = A(N,1,j)x_0.$$

We now use $\chi_q = \chi_M \chi_N$ to get

$$\begin{aligned} S_j &= \sum_{k=0}^{M-1} \chi_q(j+kN) g(A(M,1,k)v_j) \\ &= \chi_N(j) \sum_{k=0}^{M-1} \chi_M(j+kN) g(A(M,1,k)v_j). \end{aligned} \tag{4.5}$$

Let us "reduce" the points $v_1, ..., v_n$ to an approximate fundamental domain using the algorithm in [24, Chapter 7]. Hence we have matrices $\{\gamma_j, x_j\}$ such that $v_j = \gamma_j x_j$ and x_j lies in the "approximate fundamental domain".

χ_M is a character on $\mathbb{Z}/M\mathbb{Z}$. Given j, let c_j is defined by $j \equiv c_j \bmod M$. Hence rewrite (4.5) as

$$S_j = \chi_N(j) \sum_{m|M} \sum_{k=0}^{M/m-1} h_{c_j}(m,k) g(A(M,m,k)\gamma_j x_j). \tag{4.6}$$

Here for $\{0 \le l \le M-1\}$, $h_l : T(M) \to \mathbb{C}^\times$ is defined as:

$$h_l((m,k)) = \delta_1(m)\chi_M(l+kN). \tag{4.7}$$

Where, δ_1 is the characteristic function of $\{1\}$. For example, for a prime M the functions h_j are given explicitly by: $h_j(1,k) = \chi_M(j+kN)$ and $h_j(M,0) = 0$.

The number of points in $T(M)$ are at most $M^{1+\epsilon}$ (using the fact that the number of divisors of M are at most $O(M^\epsilon)$). We know that the right action by $a \in \Gamma$ permutes the $T(M)$ in $\Gamma \backslash \mathrm{SL}(2,\mathbb{R})$. In other words, given any element a of Γ, there exists a permutation σ_a on $T(M)$ such that for each $(m,k) \in T(M)$, we have an $a' \in \Gamma$ such that

$$A(M,m,k)a = a' A(M, \sigma_a(m,k)).$$

Using lemma 2.5 we get that if $a \equiv b \bmod M$, then $\sigma_a = \sigma_b$. This implies that the total number of permutations of $T(M)$ due to the right action of Γ is at most M^3. Using this, we rewrite (4.6) as

$$S_j = \chi_N(j) \sum_{(m,k) \in T(M)} h_{c_j}(\sigma_{\gamma_j}(m,k)) g(A(M,m,k)x_j). \tag{4.8}$$

Lemma 2.5 implies that the number of distinct functions $h_{c_j}\sigma_{\gamma_j}$ is at most M^4. Let us enumerate them as $r_1, ..., r_L$ (say) , where $L \le M^4$.

Notice that the Hecke orbits "preserve" the distance between the points. In other words, for any positive integer M, and any $x, y \in \mathrm{SL}(2,\mathbb{R})$ such that $x \in yV$ for some open neighbourhood V of the identity. Then $A(M,m,k)x$ lies in $A(M,m,k)yV$, for all $(m,k) \in T(M)$. Therefore, given any positive ϵ, we "sort" the points $\{x_j\}$ into sets $K_1, ..., K_Q$ such that $x, y \in K_k$ implies that $x^{-1}y \in U_{q^{-\epsilon}}$. Here $Q = O(q^{3\epsilon})$. We choose a representative w_j from each K_j.

Let us focus on K_1. Let us use a power series expansion around the point $A(M,m,k)w_1$ to compute $g(A(M,m,k)x)$ for all $x \in K_1$. We summarize the result in the following lemma:

Lemma 4.1. *Given any positive reals ϵ, γ, positive integer M, x and $y \in K_i$ for some integer i. Let r be any complex valued function defined on the the set $T(M)$. Then there are explicitly computable constants $c_{\beta,x,y}$ and d such that,*

$$J(0,r,y) = \sum_{|\beta|=0}^{d} c_{\beta,x,y} J(\beta,r,x) + O(M^2 q^{-\gamma}). \tag{4.9}$$

Here $d = O((1+\gamma)/\epsilon)$ and $J(\beta,r,x)$ is defined by

$$J(\beta,r,x) = \sum_{(m,k)\in T(M)} r((m,k)) \partial^{\beta} g(A(M,m,k)x). \tag{4.10}$$

The constant in O is independent of M, q and is a polynomial in $(1+\gamma)/\epsilon$.

We will prove the lemma 4.1 in section 5. Notice that for fixed β, r and a fixed x, we can compute $J(\beta,r,x)$ in time $O(M^{1+\epsilon})$. This gives us an exact idea of the algorithm. The exact algorithm is given by:

4.2. **Explicit algorithm.**

(1) Compute and store the functions $r_1, ..., r_L$.
(2) Reduction of points $v_1, ..., v_N$ into points $x_1, ..., x_N$ and sorting of the points into sets $K_1, ..., K_Q$ and choose a representative x_{n_i} from each K_i.
(3) For each $0 \le l < N$, compute $0 \le c_l \le M-1$ be such that $c_l \equiv l \bmod M$. Find j_l such that $h_{c_l}\sigma_{\gamma_l} = r_{j_l}$.
(4) $i \leftarrow 1$.
(5) for all x_l in K_i, and for all $|\beta| \le d$, compute and store c_{β,x_{n_i},x_l}.
(6) for all $|\beta| \le d$, and for all $1 \le t \le L$, compute and store $J(\beta, r_t, x_{n_i})$.
(7) for each x_l in K_i, compute
$$S_l = \chi_N(l) \sum_{|\beta|=0}^{d} c_{\beta,x_{n_i},x_l} J(\beta, r_{j_l}, x_{n_i},).$$
(8) if $i = Q$ compute $S = S_1 + ... + S_N$ else $i \leftarrow i+1$ and go to step 5.

4.3. **Time complexity.** Let us compute the time complexity in the algorithm. The O constants involved in this proof are polynomial in $(1+\gamma)/\eta$.

Computing and storing r_i for all i takes at most (roughly) $O(LM^{1+\epsilon})$ time. But recall that $L \le M^4$. Hence step 1 takes $O(M^{5+\epsilon})$ time.

It is easy to see that "reducing" each v_i to an approximate fundamental domain, can be done in $O(\log q)$ time. There are many standard reduction algorithms available. We refer the readers to [26] for a form of the reduction algorithm. The whole "reducing" and sorting process has also been discussed in detail in [24, chapter 7]. Therefore, the it can be easile seen that the whole reduction and sorting process in step 2 takes $O(N^{1+\epsilon})$ time . Step 3 also takes $O(N)$ time.

Notice that for each $x_l \in S_i$, we need to compute c_{β,x_{n_i},x_l} for $|\beta| \ll 1$. In section 5, it will be shown that each of these values can be computed in $O(1)$ time. Therefore, step 5 takes $O(1)$ steps. Hence for all l, step 5 takes $O(N)$ steps. Notice that for a fixed i, and fixed t, step 6 takes $O(M^{1+\epsilon})$ time.

Hence for fixed i, and for all t, step 6 takes $O(LM^{1+\epsilon})$ time. Recall that lemma 2.5 implies that $L \leq M^4$ hence total time spent in computing $J(\beta, r_t, x_{n_i})$ for a fixed i and all t is $O(M^{5+\epsilon})$. Therefore, for all i, for all j, and for all t, the step 6 takes $O(QM^{5+\epsilon}) \approx O(M^{5+4\epsilon})$ time. Recall here that $Q \approx M^{3\epsilon}$ and $d = O(1)$. Steps 7 and 8 take $O(N)$ steps.

Therefpre, the total time spent is $O(M^{5+7\epsilon} + N)$.

4.4. **Case $q = MN$ when $M|N$.** Let $q = MN$ where $M|N$. Unless redefined in this section, we borrow the notations from previous section.

We proceed as in previous section and rewrite (4.2) as

$$S = \sum_{j=0}^{N-1} S_j. \tag{4.11}$$

Here, S_j is defined as

$$S_j = \sum_{k=0}^{M-1} \chi_q(j + kN) g(A(M, 1, k)v_j). \tag{4.12}$$

v_j as in the previous section. For $(j, N) \neq 1$, $S_j = 0$. Therefore, rewrite (4.11) as

$$S_j = \sum_{j \bmod N}^{*} S_j. \tag{4.13}$$

Let $F(k) = \chi_q(1+kN)$. Since $F(k_1 + k_2) = F(k_1)F(k_2)$, there exists a constant b such that $F(k) = \chi_q(1 + kN) = e(bk/M)$. Using this we have that for any j, coprime to N,

$$S_j = \chi_q(j) \sum_{k=0}^{M-1} e(bj^{-1}k/M) g(A(M, 1, k)v_j). \tag{4.14}$$

Notice that j^{-1} denotes the multiplicative inverse of $j \bmod m$. Let c_j be defined by $c_j \equiv j \bmod M$, where $0 \leq c_j \leq q - 1$. We rewrite (4.14) as

$$S_j = \chi_q(j) \sum_{(m,k) \in T(M)}^{M/m-1} h_{c_j}((m, k)) g(A(M, m, k)v_j). \tag{4.15}$$

Here, for $0 \leq l \leq M - 1$, $h_l : T(M) \to \mathbb{C}^\times$

$$h_l((m, k)) = \delta_1(m) e(bl^{-1}k/M). \tag{4.16}$$

The function h_l only depends on the residue of $l \bmod M$. Therefore, there are M distinct functions h_{c_j}. We can proceed exactly similarly as in last section to get the required algorithm.

4.5. **Proof of theorem 1.** Let $q = MN$. Let $M = M_1M_2$ such that $(M_2, N) = 1$ and $M_1|N$. We proceed as in previous sections and rewrite (4.2) as

$$S = \sum_{j=0}^{N-1} S_j. \tag{4.17}$$

Here, S_j is defined as

$$S_j = \sum_{k=0}^{M-1} \chi_q(j+kN) g(A(M,1,k)v_j). \tag{4.18}$$

v_j as in the previous sections. Notice that $\chi_q = \chi_{M_2}\chi_{M_1N}$ and that there exists b such that $\chi_{M_1N}(1+kN) = e(kb/M_1)$. Moreover, $S_j = 0$, if j is not coprime to N. Therefore for $(j,N)=1$, rewrite (4.18) as

$$\begin{aligned} S_j &= \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2-1} \chi_q(j+(k+lM_1)N) g(A(M,1,k+lM_1)v_j) \\ &= \sum_{k=0}^{M_1-1}\sum_{l=0}^{M_2-1} \chi_q(j+kN+lM_1N) g(A(M,1,k+lM_1)v_j) \\ &= \sum_{k=0}^{M_1-1} \chi_{M_1N}(j+kN) \sum_{l=0}^{M_2-1} \chi_{M_2}(j+kN+lM_1N) g(A(M,1,k+lM_1)v_j) \\ &= \chi_{M_1N}(j) \sum_{k=0}^{M_1-1} e(\frac{bj^{-1}k}{M_1}) \sum_{l=0}^{M_2-1} \chi_{M_2}(j+kN+lM_1N) g(A(M,1,k+lM_1)v_j). \end{aligned} \tag{4.19}$$

Notice that for fixed k and l, the quantity $e(bj^{-1}k/M_1)$ is uniquely determined for $j \bmod M_1$ and $\chi_{M_2}(j+kN+lM_1N)$ is uniquely determined for $j \bmod M_2$. Hence if $j_1 \equiv j_2 \bmod M$ then for all $0 \le k \le M_1 - 1$ and $0 \le l \le M_2 - 1$ we have that,

$$e(\frac{j_1^{-1}bk}{M_1})\chi_{M_2}(j_1+kN+lM_1N) = e(\frac{j_2^{-1}bk}{M_1})\chi_{M_2}(j_2+kN+lM_1N).$$

Let c_j be defined by $c_j \equiv j \bmod M$, where $0 \le c_j \le M-1$. We can rewrite (4.19) as

$$S_j = \chi_{M_1N}(j) \sum_{(m,k)\in T(M)} h_{c_j}((m,k)) g(A(M,m,k)v_j). \tag{4.20}$$

Here, for $(l,N)=1$, $h_l : T(M) \to \mathbb{C}^\times$ is defined by:

$$h_l((m,k)) = \delta_1(m) e(bl^{-1}k/M_1)\chi_{M_2}(l+k_0N+l_0M_1N); \tag{4.21}$$

where,

$$k = k_0 + l_0M_1,\ 0 \le k_0 \le M_1 - 1,\ 0 \le l_0 \le M_2 - 1.$$

The inverse in (4.21) is modM_1. Notice again that there are only at most M distinct functions h_{c_j} on $T(M)$. Therefore, the algorithm in 4.1 canbe easily used to get an algorithm in the general case.

5. Proof of lemmas 4.1 and 2.5

In this section we will give a proof of lemmas 4.1 and 2.5.

5.1. **proof of lemma 4.1.** Let $x, y \in K_i$ for some integer i and let r be a function on the set $T(M)$. This implies that $x^{-1}y \in U_{q^{-\epsilon}}$. This implies that

$$(A(M,m,k)x)^{-1}A(M,m,k)y = x^{-1}y \in U_{q^{-\epsilon}}$$

for all $(m,k) \in T(M)$. Let

$$x^{-1}y = n(t_0)a(y_0)K(\theta_0).$$

Here $|\theta_0| \le \pi$. As g is a real analytic function on $\Gamma\backslash \mathrm{SL}(2,\mathbb{R})$, we can use the power series expansion for g at points $A(M,m,k)x$ to compute $g(A(M,m,k)y)$ to get

$$g(A(M,m,k)y) = \sum_{\beta=(\beta_1,\beta_2,\beta_3)\in\mathbb{Z}_+^3} \frac{\partial^\beta g((A(M,m,k)x)}{\beta!} t_0^{\beta_1} y_0^{\beta_2} \theta_0^{\beta_3}. \tag{5.1}$$

As $n(t_0)a(y_0)K(\theta_0) \in U_{q^{-\epsilon}}$, we get that there exists a constant $d = O((1+\gamma)/\epsilon)$ such that

$$g(A(M,m,k)y) = \sum_{|\beta|\le d} \frac{\partial^\beta g((A(M,m,k)x)}{\beta!} t_0^{\beta_1} y_0^{\beta_2} \theta_0^{\beta_3} + O(q^{-\gamma}). \tag{5.2}$$

Substituting in (4.8), we get

$$J(0,r,y) = \sum_{|\beta|=0}^{d} c_{\beta,x,y} J(\beta,r,x) + O(2M^2 q^{-\gamma}). \tag{5.3}$$

Here $c_{\beta,x,y} = t_0^{\beta_1} y_0^{\beta_2} \theta_0^{\beta_3}/\beta!$. Notice that each $c_{\beta,l,n}$ can be computed in unit time.

5.2. **proof of lemma 2.5.** Let (m,k) be a any element of $T(L)$. Let $\sigma_{a_1}(m,k) = (m_1,k_1)$. This implies that $\Gamma A(L,m,k)a_1 = \Gamma A(L,m_1,k_1)$. This implies that there exist a matrix $c \in \mathrm{SL}(2,\mathbb{Z})$ such that

$$cA(L,m,k)a_1 = A(L,m_1,k_1). \tag{5.4}$$

The lemma is equivalent is proving that there exists a matrix d in $\mathrm{SL}(2,\mathbb{Z})$, such that

$$dA(L,m,k)a_2 = A(L,m_1,k_1) \tag{5.5}$$

(this would imply that $\sigma_{a_2}(m,k) = \sigma_{a_1}(m,k)$).

We use (5.4) to get

$$\begin{aligned}
& cA(L,m,k)a_2 \\
&= cA(L,m,k)(a_1 + Lb) \\
&= cA(L,m,k)a_1 + cA(L,m,k)Lb \\
&= A(L,m_1,k_1) + cA(L,m,k)Lb \\
&= A(L,m_1,k_1) + cA(L,m,k)LbA(L,m_1,k_1)^{-1}A(L,m_1,k_1) \\
&= (I + cA(L,m,k)LbA(L,m_1,k_1)^{-1})A(L,m_1,k_1) \\
&= (I + cL^{-\frac{1}{2}} \begin{pmatrix} m & k \\ 0 & L/m \end{pmatrix} LbL^{-\frac{1}{2}} \begin{pmatrix} L/m_1 & -k_1 \\ 0 & L/m_1 \end{pmatrix})A(L,m_1,k_1) \\
&= (I + c \begin{pmatrix} m & k \\ 0 & L/m \end{pmatrix} b \begin{pmatrix} L/m_1 & -k_1 \\ 0 & L/m_1 \end{pmatrix})A(L,m_1,k_1). \qquad (5.6)
\end{aligned}$$

(5.6) implies that $d = c(I + c\begin{pmatrix} m & k \\ 0 & L/m \end{pmatrix} b \begin{pmatrix} L/m_1 & -k_1 \\ 0 & L/m_1 \end{pmatrix})^{-1}$ will be in $\mathrm{SL}(2,\mathbb{Z})$ and will satisfy condition in equation (5.5). (m,k) was any arbitrary element of $T(L)$. This implies that $\sigma_{a_1} = \sigma_{a_2}$.

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