

Classification of critical points

The test

Let (a, b) be a critical point of $f(x, y)$. Let $P = f_{xx}(a, b)$, $Q = f_{xy}(a, b)$ and $R = f_{yy}(a, b)$.

1. If $PR - Q^2 < 0$, then (a, b) is a saddle point.
2. If $PR - Q^2 > 0$ and $P > 0$, then (a, b) is a local minimum.
3. If $PR - Q^2 > 0$ and $P < 0$, then (a, b) is a local maximum.
4. If $PR - Q^2 = 0$, no conclusion can be reached without further evidence.

Why this works

If (a, b) is a critical point of $f(x, y)$, then $f_x(a, b) = 0 = f_y(a, b)$. Near any such point, Taylor expansion gives, with P , Q and R as above,

$$f(a + h, b + k) = f(a, b) + \frac{1}{2} (Ph^2 + 2Qhk + Rk^2) + \text{terms of higher order in } h \text{ and } k.$$

For sufficiently small h and k , the sign of the difference $f(a + h, b + k) - f(a, b)$ is that of the quadratic $Ph^2 + 2Qhk + Rk^2$. The critical point is a local maximum if this is negative for all values of h and k , a local minimum if it is positive for all values of h and k , and a saddle point if it is negative for some values h and k and positive for others.

If $P \neq 0$ we can write the quadratic as

$$\begin{aligned} Ph^2 + 2Qhk + Rk^2 &= \frac{1}{P} (P^2h^2 + 2PQhk + PRk^2) \\ &= \frac{1}{P} [(Ph + Qk)^2 + (PR - Q^2)k^2]. \end{aligned}$$

If $PR - Q^2 > 0$ the whole quadratic has the same sign as P , so it is positive if $P > 0$ and negative if $P < 0$. If $PR - Q^2 < 0$ the whole expression is positive for some values of h and k , and negative for others. Finally, if $PR - Q^2 = 0$ the quadratic vanishes for those values of h and k for which $Ph + Qk = 0$; then the sign of $f(a + h, b + k) - f(a, b)$ is determined by terms of higher order, which are not considered here.

So far we have assumed that $P \neq 0$. If $P = 0$ and $Q \neq 0$, then $PR - Q^2 < 0$. In that case the quadratic is not of constant sign, in agreement with item 1 of the test. Finally, if P and Q are both zero, item 4 of the test applies; then a conclusion would depend on the sign of R or, if $R = 0$, information about terms of higher order.