

Laplace's equation in polar coordinates

If V is a function of x and y , where $x = r \cos \theta$ and $y = r \sin \theta$, we can show that

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}.$$

By the chain rule,

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial V}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial V}{\partial x} + \sin \theta \frac{\partial V}{\partial y}.$$

We can write

$$\frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y},$$

with the idea of applying this to V or to any other function we want to differentiate. Using this, partial differentiation again with respect to r gives

$$\begin{aligned} \frac{\partial^2 V}{\partial r^2} &= \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial V}{\partial x} \right) + \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial V}{\partial y} \right) \\ &= \cos \theta \left(\cos \theta \frac{\partial^2 V}{\partial x^2} + \sin \theta \frac{\partial^2 V}{\partial y \partial x} \right) + \sin \theta \left(\cos \theta \frac{\partial^2 V}{\partial x \partial y} + \sin \theta \frac{\partial^2 V}{\partial y^2} \right) \\ &= \cos^2 \theta \frac{\partial^2 V}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 V}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 V}{\partial y^2}. \end{aligned}$$

Here I have used the equality of the mixed partial derivatives.

Similarly, partial differentiation with respect to θ gives

$$\frac{\partial V}{\partial \theta} = -r \sin \theta \frac{\partial V}{\partial x} + r \cos \theta \frac{\partial V}{\partial y},$$

and we can write

$$\frac{\partial}{\partial \theta} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}.$$

A second partial differentiation gives

$$\begin{aligned} \frac{\partial^2 V}{\partial \theta^2} &= -r \left(\cos \theta \frac{\partial V}{\partial x} + \sin \theta \frac{\partial}{\partial \theta} \frac{\partial V}{\partial x} \right) + r \left(\cancel{-\sin \theta \frac{\partial V}{\partial x}} + \cos \theta \frac{\partial}{\partial \theta} \frac{\partial V}{\partial y} \right) \\ &= -r \frac{\partial V}{\partial r} - r \sin \theta \left(-r \sin \theta \frac{\partial^2 V}{\partial x^2} + r \cos \theta \frac{\partial^2 V}{\partial y \partial x} \right) \\ &\quad + r \cos \theta \left(-r \sin \theta \frac{\partial^2 V}{\partial x \partial y} + r \cos \theta \frac{\partial^2 V}{\partial y^2} \right) \\ &= -r \frac{\partial V}{\partial r} + r^2 \left(\sin^2 \theta \frac{\partial^2 V}{\partial x^2} - 2 \sin \theta \cos \theta \frac{\partial^2 V}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 V}{\partial y^2} \right). \end{aligned}$$

Combining these results we get the required relationship.