1 Functions of several variables

Graphs of functions

An equation of the form $z = f(x, y)$ thus describes a surface. We can think of z as the height of the surface above the x, y plane. (or below if it is negative).

2 Partial derivatives

For any function $f(x, y)$ we get

$\left(\frac{\partial f}{\partial x}\right)_y$ by differentiating w.r.t. x keeping y constant

$\left(\frac{\partial f}{\partial y}\right)_x$ by differentiating w.r.t. y keeping x constant

We often drop the subscripts . Furthermore, sometimes $\frac{\partial f}{\partial x}$ is written f_x and $\frac{\partial f}{\partial y}$ is written f_y.

Differentiating again we get

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ also written f_{xx}

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ also written f_{yx}

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$ also written f_{xy}

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ also written f_{yy}

For all "sensible" functions $f_{xy} = f_{yx}$.

Mathematical definition of the partial derivative

Mathematical definition of the partial derivative is similar:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Recall that $\frac{df}{dx}$ gives the rate of change of f. Similarly $\frac{\partial f}{\partial x}$ corresponds to the rate of change of f **in the positive x direction**. And $\frac{\partial f}{\partial y}$ corresponds to the rate of change of f **in the positive y direction**.

Multi-variable partial derivatives

Partial derivatives are defined for multi-variable functions in a similar way: For a function of n variables, $f(x_1, x_2, \ldots, x_n)$, we get:

$\frac{\partial f}{\partial x_i}$ by differentiating w.r.t. the variable x_i keeping **all other variables** constant

Mathematically we define

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{\Delta x_i}$$

The total differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

We call df, defined by this equation, the **total differential** of f.

Exact and Inexact differentials

Not all differentials are total differentials of a function. For example $y\,dx + x\,dy$ is the total differential df of the function $f(x, y) = xy + c$ where c is a constant (**Check this!**). However try to find a function $f(x, y)$ whose total differential is $x\,dy + 3y\,dx$. It is impossible, there is no such function. Such a differential which is not the total differential of any function is known as an **inexact differential.** On the other hand a function which is the total differential of a function is known as an **exact differential.**

A differential $A(x, y)\,dx + B(x, y)\,dy$ is exact if and only if $A_y = B_x$.

The multi-variable case

This works similarly. The **total differential** of a multi-variable function $f(x_1, x_2, \ldots, x_n)$ is

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

A differential

$$\sum_{i=1}^{n} g_i(x_1, x_2, \ldots, x_n) dx_i$$

is **exact** if it is the **total differential** of a function, ie if $g_i = \partial f / \partial x_i$ for some function f, and inexact otherwise. A necessary and sufficient condition for a differential to be exact is:

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}.$$

For all $i, j = 1, \ldots, n$.

The chain rule

Multi variable case: If $f = f(x, y)$ and x and y and both function of just t then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Change of variables

Suppose that $x = x(u, v)$ and $y = y(u, v)$ where u, v are two other variables. Then we have:

$$\begin{aligned}
\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\
\frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}
\end{aligned}$$

Note that $\frac{\partial f}{\partial x}$ is at constant y but $\frac{\partial x}{\partial u}$ is at constant v.

3 Partial differential equations

An equation involving *functions of two or more variables and some of its spatial derivatives* is a p.d.e. Here we solve some simple examples that can be done easily.

Simple Examples

Example 4: Find the general solution $z(x,y)$ of $\frac{\partial^2 z}{\partial x \partial y} = x - y$.

Integration w.r.t x keeping y constant gives $z_y = \frac{x^2}{2} - xy + G(y)$. And with respect to y gives

$$z = \frac{xy}{2}(x - y) + f(x) + g(y)$$

where f and g are again arbitrary differentiable functions.

Example 6: Find the general solution $u(x,y)$ of $u_{xx} + \frac{u_x}{x} = 3x + 4$.

Let $p(x,y) = u_x$ then $xp_x + p = 3x^2 + 4x$ can be integrated straightaway

$$\begin{aligned}
xp &= x^3 + 2x^2 + f(y) \\
u_x &= x^2 + 2x + \frac{f(y)}{x}
\end{aligned}$$

and then integrating again gives

$$u = \frac{x^3}{3} + x^2 + f(y)\log(x) + g(y).$$

More interesting examples

Example 9: The wave equation $u_{xx} - c^{-2}u_{tt} = 0$.

This is the type of equation describing the string, and many other systems. The way to solve this problem can be seen by rewriting it. Then again following example 8 by the chain rule

$$\begin{aligned}
\partial_x u &= \partial_s u \partial_x s + \partial_r u \partial_x r = \partial_s u + \partial_r u \\
\partial_t u &= \partial_s u \partial_t s + \partial_r u \partial_t r = c\partial_s u - c\partial_r u
\end{aligned}$$

Then the equation becomes

$$u_{rs} = 0.$$

Integrating as in example 3 gives $u = f(r) + g(s) = f(x + ct) + g(x - ct)$. This is d'Alembert's solution.

This is the sum of a left moving wave with speed c and a right moving wave.

4 Taylor expansions (Riley 5.7)

This last equation makes it obvious how to generalize; a function of two variables expanded about $(x, y) = (x_0, y_0)$ can be found by first expanding about $x = x_0$ and then about $y = y_0$;

$$f(x, y) = f(x_0 + \Delta x, y_0 + \Delta y) \quad = \quad f + \Delta x\, f_x| + \Delta y\, f_y| + \frac{1}{2}(\Delta x^2 f_{xx} + 2\Delta x \Delta y f_{xy} + \Delta y^2 f_{yy})| +$$

where the vertical line | indicates setting $x \to x_0$ and $y \to y_0$.

5 Critical points

Want to find where local maxima minima or saddle points are. A critical point is a point at which both $f_x = 0$ and $f_y = 0$.

Local maxima and minima

Defn: A point (x_0, y_0) is said to be a local *maximum* if $f(x_0, y_0) > f(x, y)$ for all points (x, y) in a sufficiently small neighbourhood surrounding (x_0, y_0).

Defn: A point (x_0, y_0) is said to be a local *minimum* if $f(x_0, y_0) < f(x, y)$ for all points (x, y) in a sufficiently small neighbourhood surrounding (x_0, y_0).

A necessary condition for a maximum, minimum or saddle is that $f_x = f_y = 0$.

The test for what sort of critical point it is is let $M = f_{xx}f_{yy} - (f_{xy})^2$

- If $M > 0$ and $f_{xx} > 0$ then local minimum

- If $M > 0$ and $f_{xx} < 0$ then local maximum

- If $M < 0$ then saddle point

- If $M = 0$ then inconclusive

Proof: the function near x_0, y_0 is

$$\begin{aligned}
f(x_0 + \Delta x, y_0 + \Delta y) &= f(x_0, y_0) + \frac{1}{2f_{xx}}(\Delta x^2 f_{xx}^2 + 2\Delta x \Delta y \, f_{xy}f_{xx} + \Delta y^2 f_{yy}f_{xx}) + \dots \\
&= f(x_0, y_0) + \frac{1}{2f_{xx}}((\Delta x \, f_{xx} + \Delta y \, f_{xy})^2 + \Delta y^2 M) + \dots .
\end{aligned}$$

If $f_{xx} > 0$ and $M > 0$ then $f(x,y) - f(x_0, y_0) > 0$ and $f(x_0, y_0)$ is a minimum. If $f_{xx} < 0$ then the reverse is true. If $M < 0$ then for some values of $\Delta x, \Delta y$, $f(x,y) - f(x_0, y_0)$ is positive and for others it's negative.

6 Double integrals (Riley 6.1)

Interpretation as volume under a surface

- **An important example** of double integration region is when $f(x, y) = 1$; the "volume" under this surface is clearly just **the area of the integration region**;

$$V = \iint_R dA = A.$$

Double integrals over regions

Suppose that instead the region R is defined by

$$\begin{aligned}
a &\leq x \leq b \\
u(x) &\leq y \leq v(x)
\end{aligned}$$

where u, v are some functions. Integration becomes

$$\iint_R f(x,y)dA = \int_a^b \int_{u(x)}^{v(x)} f(x,y)dy \, dx.$$

Change of variables in double integration (Riley 6.4.1)

$J = |x_u y_v - x_v y_u|$ is known as the Jacobian and is sometimes denoted

$$\frac{\partial(x,y)}{\partial(u,v)} = J = |x_u y_v - x_v y_u|.$$

$$\iint_R f(x,y)dxdy = \iint_R f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)} dudv = \iint_R f(x(u,v), y(u,v))|x_u y_v - x_v y_u|dudv$$

Use of polar coordinates

Here $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = |x_r y_\theta - x_\theta y_r| = |\cos\theta(r\cos\theta) - (-r\sin\theta)\sin\theta| = r$$

and so again from the previous subsection the area element $dA = J\,dr\,d\theta = r\,dr\,d\theta$.

7 Extension to triple integration

The total mass can be written as the triple integral

$$M = \iiint_V \rho\,dV$$

where V is the integration volume.

Cylindrical polar coordinates

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_V f(r\cos\phi, r\sin\phi, z)\,rdrd\phi dz$$

Applications: volumes, masses, centres of masses and centroids

Masses and volumes already seen. Centre of mass of a body has coordinates $\bar{x}, \bar{y}, \bar{z}$ where

$$\bar{x}\int \rho\,dV = \int x\,\rho\,dV$$

and similar for y, z.

Definition: Centroid= what the centre of mass of an object would be if it had constant density (even if it doesn't.) (Use above formula with $\rho = 1$)

Change of variables for triple integrals: Spherical polar coordinates

$$dV = dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Cylindrical polar coordinates

Here we have coordinates (r, ϕ, z) where $x = r\cos\phi, y = r\sin\phi, z = z$ so the Jacobian becomes

$$\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

as we saw already.

Spherical polar coordinates

The coordinates x, y, z are given by

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta. \end{aligned}$$

The Jacobian is then

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix} = r^2\sin\theta$$

$$\iiint_V f(x,y,z)\,dxdydz = \iiint_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, z\cos\theta)\,r^2\sin\theta\,drd\phi d\theta$$

8   Vector Calculus

Combine together what we know about vectors with what we know about calculus. Already seen this a little bit...

Revision of 1st term: vectorial functions of one variable

In A-level: scalar functions of one variable. ODEs: Physical picture, particle moving in 1 dimension $x(t)$. Vectorial function of 1 variable. Picture: Particle traveling in 3d $\mathbf{r}(t)$

Basic formula: (here the variable is t)

$$\frac{d\mathbf{a}(t)}{dt} = \frac{da_1}{dt}\mathbf{i} + \frac{da_2}{dt}\mathbf{j} + \frac{da_3}{dt}\mathbf{k}$$

Chain rule:

$$\frac{d\mathbf{a}(s)}{du} = \frac{ds}{du}\frac{d\mathbf{a}}{ds}$$

Non-constant basis, eg Polar coordinates: (See term 1 handout)

Often even the basis vectors need to be differentiated. Cartesians $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant, but in Polars, for example, they are not constant. Quick recap.

Consider motion in a plane, using polar coordinates r, θ, where $x = r\cos\theta$ and $y = r\sin\theta$. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$.

The **radial unit vector** $\mathbf{e}_r$ is a vector in the direction of $\mathbf{r}$, $\mathbf{e}_r = \frac{\mathbf{r}}{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

The **tangential unit vector** $\mathbf{e}_\theta$ is a vector perpendicular to $\mathbf{e}_r$, and is $\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, (increasing θ is anti-clockwise).

If the particle is moving then r and θ can depend on time. $\dot{\mathbf{e}}_r = \frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{e}_r = \dot\theta(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot\theta\mathbf{e}_\theta$. $\dot{\mathbf{e}}_\theta = \frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{e}_\theta = \dot\theta(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = -\dot\theta\mathbf{e}_r$.
Note: $\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1$ and $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$ for all time.
$\mathbf{r} = r\mathbf{e}_r$ therefore $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot\theta\mathbf{e}_\theta$. $\ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot\theta\mathbf{e}_\theta + r\ddot\theta\mathbf{e}_\theta + r\dot\theta\dot{\mathbf{e}}_\theta = (\ddot{r} - r\dot\theta^2)\mathbf{e}_r + (2\dot{r}\dot\theta + r\ddot\theta)\mathbf{e}_\theta$.

8.1 Integration of vectors

(Indefinite) Integration of vectors (or expressions involving vectors) wrt scalar = inverse of differentiation. But rememember: if the expression we integrate is a vector, then

- the integral is also a vector

- constant of integration is also a vector

eg if $\mathbf{a}(t) = d\mathbf{A}/dt$, then the indefinite integral of $\mathbf{A}$ is

$$\int \mathbf{a}(t)\, dt = \mathbf{A}(t) + \mathbf{b}$$

where $\mathbf{b}$ is a constant vector (since $\mathbf{a}(t)$ is a vector.) As you would expect the definite integral from $t = t_1$ to $t = t_2$ is

$$\int_{t_1}^{t_2} \mathbf{a}(t)\, dt = \mathbf{A}(t_1) - \mathbf{A}(t_2)$$

Vector functions of several arguments

$\mathbf{a}(u_1, u_2, \ldots, u_n)$ vector function of n variables.

Basic formula still applies

$$\frac{\partial \mathbf{a}}{\partial u_r} = \frac{\partial a_1}{\partial u_r}\mathbf{i} + \frac{\partial a_2}{\partial u_r}\mathbf{j} + \frac{\partial a_3}{\partial u_r}\mathbf{k}$$

Chain rule generalises to the multi-variable case

If $\mathbf{a}(u_1, u_2, \ldots, u_n)$ and each $u_i(v_1, v_2, \ldots, v_n)$, then just as for scalar functions we have

$$\frac{\partial \mathbf{a}}{\partial v_i} = \frac{\partial \mathbf{a}}{\partial u_1}\frac{\partial u_1}{\partial v_i} + \frac{\partial \mathbf{a}}{\partial u_2}\frac{\partial u_2}{\partial v_i} + \cdots + \frac{\partial \mathbf{a}}{\partial u_n}\frac{\partial u_n}{\partial v_i}$$

Total differential generalises straightforwardly

If we have a vector function of n variables, $\mathbf{a}(u_1, u_2, \ldots, u_n)$ then the total derivative is:

$$d\mathbf{a} = \frac{\partial \mathbf{a}}{\partial u_1}du_1 + \frac{\partial \mathbf{a}}{\partial u_2}du_2 + \cdots + \frac{\partial \mathbf{a}}{\partial u_n}du_n.$$

Scalar and Vector fields

A scalar field is simply a function $f(x, y, z)$ where x, y, z are co-ordinates of space

A vector field is a vectorial function of x, y, z. So it associates a **vector** $\mathbf{v}(x, y, z)$ to every point in space (x, y, z).

Vector Operators: Grad

Summary of Grad

- In Cartesian coordinates: $\nabla f = (\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k})$

- Takes a scalar field $f(x, y, z)$ to a vector field ∇f

- ∇f points in the direction of maximum gradient (Grad is short for the "gradient operator".)

- Directional derivative: The rate of change in the direction $\hat{\mathbf{n}}$ (unit vector) is $\hat{\mathbf{n}} \cdot \nabla f$

- Level curves (in 2d - or level surfaces in 3d) is the curves $f(x, y, z) = c$ are perpendicular to ∇f.

Vector Operators: Div

The divergence of a vector field $\mathbf{v}(x, y, z) = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$, "Div $\mathbf{v}$" is defined by

$$\operatorname{div}\mathbf{v} = \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

Significance of Div: If we think of the vector field $\mathbf{v}(x, y, z)$ as giving the flow of some quantity, then $\nabla \cdot \mathbf{v}$ gives a measure of the net amount flowing out of any point.

Laplacian: $\nabla^2 f$

$$\operatorname{div}(\operatorname{grad}f) = \nabla \cdot (\nabla f) = \nabla \cdot (f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}) = f_{xx} + f_{yy} + f_{zz}$$

Div(Grad) $= \nabla^2$ is the Laplacian (which we met previously). (Can be defined in any dimension.)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Vector Operators: Curl

The operator **curl** only works in 3 dimensions. It is defined as follows, for any vector field $\mathbf{v}(x, y, z) = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}$:

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}.$$

This is, of course what you get from $\nabla \times \mathbf{v} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$

Curl gives a measure of the angular velocity of the fluid near a point.

Famous example of Grad, Div, Curl

Maxwell's equations for Electro-magnetism. **E** is the electric field (it is a vector field) **B** is the magnetic field (it is also a vector field). **B** points in the direction of a compass. In a vacuum these are governed by Maxwell's equations which are written in terms of Grad, Div and Curl:

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

More later.....

Vector operator Formulae [Riley:10.8]

Vector operators (Grad, Div, Curl) acting on Sums/products

1. Grad, Div, Curl are **linear operators** (means "sum then operate=operate then sum") ie:

$$\nabla(f + g) = \nabla f + \nabla g$$
$$\nabla \cdot (\mathbf{v} + \mathbf{w}) = \nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{w}$$
$$\nabla \times (\mathbf{v} + \mathbf{w}) = \nabla \times \mathbf{v} + \nabla \times \mathbf{w}$$

2. When acting on products involving two scalar fields or a scalar field and a vector field we get natural generalisations of the product rule:

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$
$$\nabla \cdot (f\mathbf{v}) = (\nabla f) \cdot \mathbf{v} + f(\nabla \cdot \mathbf{v})$$
$$\nabla \times (f\mathbf{v}) = (\nabla f) \times \mathbf{v} + f(\nabla \times \mathbf{v})$$

3. Vector operators acting on products of vector fields can be trickier. The first is the simplest as there are only two terms:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = (\nabla \times \mathbf{a}) \cdot \mathbf{b} - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

Then we also have the following more complicated identities

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}) + (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a}$$
$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

Action on the position vector and related functions

We write the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. Then we have for example

$$\nabla \cdot \mathbf{r} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 3$$

and since $\partial r/\partial x = x/r$ etc. we have

$$\nabla r = (\frac{x}{r})\mathbf{i} + (\frac{y}{r})\mathbf{j} + (\frac{z}{r})\mathbf{k} = \frac{\mathbf{r}}{r} = \mathbf{e}_r.$$

Combinations of Grad, Div and Curl

We can also consider applying more than one of Grad, Div or Curl on a single scalar or vector field. We get two identities that give zero

$$\mathrm{curl}(\mathrm{grad}f) = \nabla \times (\nabla f) = 0$$
$$\mathrm{div}(\mathrm{curl}\,\mathbf{v}) = \nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\mathrm{curl}(\mathrm{curl}\,\mathbf{v}) = \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v}$$

13

where we recall that $\nabla^2 f = \nabla \cdot \nabla f$ (although here it is applied to a vector field.)

General Curvilinear Coordinates

Instead of x, y, z it is sometimes useful to use different co-ordinates u_1, u_2, u_3 (eg Spherical or Cylindrical Polar co-ordinates) so that

$$x = x(u_1, u_2, u_3) \qquad y = y(u_1, u_2, u_3) \qquad z = z(u_1, u_2, u_3)$$

Basis vectors

We define basis vectors:

$$\mathbf{e_1} = \frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial u_1} \qquad \mathbf{e_2} = \frac{1}{h_2} \frac{\partial \mathbf{r}}{\partial u_2} \qquad \mathbf{e_3} = \frac{1}{h_3} \frac{\partial \mathbf{r}}{\partial u_3}$$

where h_1.... are called *scale factors:*

$$h_1 = \left| \frac{\partial \mathbf{r}}{\partial u_1} \right| \qquad h_2 = \left| \frac{\partial \mathbf{r}}{\partial u_2} \right| \qquad h_3 = \left| \frac{\partial \mathbf{r}}{\partial u_3} \right|$$