Some important PDEs

1. *The wave equation*

   In one dimension,
   $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$
   The general solution is $u(x,t) = f(x+ct)+g(x-ct)$, where f and g are arbitrary twice-differentiable functions.

   (a) Small transverse oscillations of a stretched string, e.g., on a guitar, violin, piano, etc. Then $u(x,t)$ is the displacement of the string from its equilibrium position, at a distance x from one end of the string at time t. In that case, $c^2 = T/\rho$, where T is the tension in the string and ρ is its density (mass per unit length).

   (b) Longitudinal waves (sound waves) in a straight column of air, as in an organ pipe, flute, recorder, etc. Then c is the speed of sound in air, and u is the displacement of a particle of air from its equilibrium position.

   (c) Torsional oscillations of a bar, such as a drive-shaft of a machine. Then u is the angle of twist at a distance x from an end of the shaft at time t, and c depends on properties of the material from which the shaft is made.

   In two dimensions,
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$
   This provides a mathematical model for small oscillations of a membrane, such as a drum skin.

   In three dimensions,
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$
   This provides a model for the propagation of sound waves, in which case c is the speed of sound in the relevant medium—air, water etc. It also describes electromagnetic waves—light, radio and television waves, radar, microwaves, etc.—and then c is the speed of light.

   In any number of dimensions the wave equation may be written as
   $$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},$$
   where $\nabla^2 u$, read as "del-squared u", stands for the left-hand side of the wave equation in the appropriate number of dimensions. The notation comes from thinking of the dot product of the gradient operator with itself. In three dimensions,
   $$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$
   The expression $\nabla^2 u$ is also called the Laplacian of u. Sometimes symmetry considerations make other coordinate systems preferable to the rectangular cartesian coordinates x, y and z.

   In cylindrical polar coordinates, r and θ and z, with $x = r\cos\theta$ and $y = r\sin\theta$,
   $$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}.$$
   The form we found for plane polar coordinates is obtained by removing the dependence on z.

   In spherical polar coordinates, r, θ and ϕ, where $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$,
   $$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

2. *The diffusion equation (heat equation)*

   In one dimension,

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{D}\frac{\partial C}{\partial t},$$

   where $C(x,t)$ is the concentration of the diffusing substance, and the constant D is called the *diffusivity* or *diffusion coefficient*. In two or three dimensions the second partial derivative with respect to x is replaced by $\nabla^2 C$.

   There are many examples of diffusion processes.

   (a) A drop of ink or other coloured liquid placed in water gradually spreads out.

   (b) Oxygen absorption into the blood stream in the lungs, and elimination of carbon dioxide.

   (c) Diffusion of sodium and potassium ions through the walls of a nerve axon plays an important role in the propagation of nerve impulses along the axon.

   (d) Heat flow is a diffusion process. In that case C is the temperature.

   Reaction-diffusion equations have the form

$$D\nabla^2 C = \frac{\partial C}{\partial t} + f(C),$$

   where the function f represents some process generating the diffusing substance. For example, neutrons generated by radioactive decay in a lump of uranium or plutonium diffuse out of the lump and can also trigger the release of further neutrons. In a sufficiently large lump, diffusion is not fast enough to prevent a chain reaction and, if that is not controlled, a nuclear explosion.

3. *Laplace's equation*

   If we remove the time-dependence from the wave equation or the diffusion equation we get Laplace's equation,

$$\nabla^2 u = 0.$$

   This describes a variety of steady-state situations. It gives the equilibrium temperature distribution in a heat-flow problem, and the equilibrium concentration in other diffusion problems. It is satisfied by the electrostatic potential in uncharged regions, and by the gravitational potential outside any distributions of matter.

   A closely related equation is Poisson's equation,

$$\nabla^2 u = \rho(\mathbf{r}),$$

   where $\mathbf{r}$ is the position vector of a point and ρ is a given function. The gravitational potential within the material of a star or planet satisfies Poisson's equation.

4. *The Schrödinger equation*

   The time-independent Schrödinger equation, in suitable units, is

$$-\nabla^2\psi + V\psi = E\psi.$$

   Here V is a given potential function, E is a constant energy, and the solution ψ is called the wave function of the problem.