SINGLE MATHS B - VECTORS - SUMMARY

Definition and notations

Definition: A vector is a directed line segment.

Abstractly write as $\mathbf{a}$ or as $\overrightarrow{PQ}$

Quantitively write e.g.

$\mathbf{a} = (2, 2, 1) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

Addition and Scalar Multiplication

Addition: use the tip to tail rule

Addition in components

$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Multiplication by a scalar: multiplies length of vector

Multiplication by a scalar in components

$s\mathbf{a} = (sa_1, sa_2, sa_3)$

Parallel: $\mathbf{a}$ and $\mathbf{b}$ are parallel means

$\mathbf{a} = s\mathbf{b}$ for some number s

Dot(Scalar) Product

"vector $\times$ vector = scalar"

Geometrically: $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

In components: $(a_1, a_2, a_3).(b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3$

Length of a vector: $|\mathbf{a}| = \sqrt{\mathbf{a}.\mathbf{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Perpendicular: $\mathbf{a}$ and $\mathbf{b}$ are perpendicular means $\mathbf{a}.\mathbf{b} = 0$

Component in direction of unit vector $\hat{\mathbf{n}}$: $d = \mathbf{a}.\hat{\mathbf{n}}$

Vector(Wedge) Product

"vector$\times$ vector = vector"

Geometrically: $\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \hat{\mathbf{n}}$

$\hat{\mathbf{n}}.\mathbf{a} = \hat{\mathbf{n}}.\mathbf{b} = 0$, $|\hat{\mathbf{n}}| = 1$

$\hat{\mathbf{n}}$ points in direction a screw would go if screwed from $\mathbf{a}$ to $\mathbf{b}$.

In components:

$$(a_1, a_2, a_3) \wedge (b_1, b_2, b_3) = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Area of parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \wedge \mathbf{b}|$

Triple products

Scalar triple product: $\mathbf{a}.(\mathbf{b} \wedge \mathbf{c})$

Volume of box with sides $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{a}.(\mathbf{b} \wedge \mathbf{c})$

Vector triple product: $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$.

Note that $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \neq (\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c}$

Lines and Planes

Equation of line through the point A with direction $\mathbf{b}$

$\mathbf{x}$(t)=$\mathbf{a}$+t$\mathbf{b}$

Equation of a plane containing $\mathbf{b}$ and $\mathbf{c}$ through the point A

$\mathbf{x}$(s,t)=$\mathbf{a}$+s$\mathbf{b}$+t$\mathbf{c}$

Cartesian equation of plane: $\mathbf{x}.\hat{\mathbf{n}} = d$

where $\hat{\mathbf{n}}$ is normal to the plane and d is distance from origin

Differentiation by scalars

Differentiate component by component: $\frac{d\mathbf{a}}{dt} = (\frac{da_1}{dt}, \frac{da_2}{dt}, \frac{da_2}{dt})$

Leibnitz rule:

$$\frac{d}{dt}(s\mathbf{a}) = \frac{ds}{dt}\mathbf{a} + s\frac{d\mathbf{a}}{dt}$$

$$\frac{d}{dt}(\mathbf{a}.\mathbf{b}) = \frac{d\mathbf{a}}{dt}.\mathbf{b} + \mathbf{a}.\frac{d\mathbf{b}}{dt}$$

$$\frac{d}{dt}(\mathbf{a} \wedge \mathbf{b}) = \frac{d\mathbf{a}}{dt} \wedge \mathbf{b} + \mathbf{a} \wedge \frac{d\mathbf{b}}{dt}$$

Mechanics

For a particle with displacement $\mathbf{x}(t)$

Velocity: $\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}$

Momentum: $\mathbf{p} = m\mathbf{v}$ where m is the mass

Acceleration: $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{x}(t)}{dt^2}$

Newtons second Law: Force $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

Torque: $\mathbf{T} = \mathbf{x} \wedge \mathbf{F}$

Angular momentum: $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$

Angular Newton's Law: $\mathbf{T} = \frac{d\mathbf{L}}{dt}$

Polar coordinates

2-dimensional polar coordinates: $\mathbf{x} = r\mathbf{e}_r$

In Cartesians: $\mathbf{e}_r = (\cos\theta, \sin\theta)$, $\mathbf{e}_\theta = (-\sin\theta, \cos\theta)$

Velocity in polars: $\dot{\mathbf{x}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$

Acceleration in polars:$\ddot{\mathbf{x}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$

3-dimensional polar coordinates

Cylindrical polars:(r, θ, z)

Spherical polars:(r, θ, ϕ)