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Topics in Combinatorics IV, Solutions 1 (Week 1)

1.1. ($\star$) Compute the number of Dyck paths of length $2n$ which start with two steps up.

Solution: Let us compute the number of Dyck paths that *do not* have the required property. Equivalently, we want to compute the number of Dyck paths whose second step is down. Then there is a clear bijection between the set of such paths and the set of Dyck paths of length $2n-2$ (just consider the path from $(2,0)$ to $(2n,0)$), so the answer is $C_n - C_{n-1}$.

1.2. ($\star$)

(a) Let P be a Dyck path of length $2n$, let $a_1,\dots,a_n$ be the positions of the steps "down", $1 \le a_i \le 2n$. Show that $a_i \ge 2i$ for every $i \le n$.

(b) Show that the number of strictly increasing sequences $(a_1,\dots,a_n)$ of integers satisfying $2i \le a_i \le 2n$ is the n-th Catalan number C_n.

(c) Show that the number of strictly increasing sequences $(a_1,\dots,a_{n-1})$ of integers satisfying $1 \le a_i \le 2i$ is the n-th Catalan number C_n.

Solution:

(a) Before i-step down one must have $i-1$ steps down and at least i steps up, so $a_i > 2i-1$ as required.

(b) Consider a lattice path where on every a_i-th place the paths goes down, and on the others it goes up. The assumptions on the sequence guarantee that we obtain a Dyck path of length $2n$: the computation made in the proof of (a) says that for every place a_i the number of steps up before is is always larger than the number of steps down. Moreover, positions of steps down define a Dyck path uniquely, so this is an injective map. The inverse is given by (a).

(c) The bijection with sequences in (b) can be constructed as follows: take the "complementary" sequence, i.e the sequence of positions of steps up, remove the first entry (which is always 1), and subtract 1 from each term.

1.3. Show explicitly that the number of triangulations of an $(n+2)$-gon satisfies the Catalan recursion (see Lemma 1.10 from lectures).

Solution: Denote the number of triangulations of an $(n+2)$-gon by T_n, put $T_0 = 1$. Fix a vertex of the $(n+2)$-gon, call it 0, and index vertices counterclockwise from 0 to $n+1$. Take any triangulation, and consider the minimal k which is connected to 0 by a diagonal.

If there is no such vertex, then there is a triangle $(0,1,n+1)$, and the number of such triangulations is equal to the number of all triangulations of the $(n+1)$-gon $(1,\dots,n+1)$, i.e. T_{n-1}.

Otherwise, the diagonal between 0 and k (note that $2 \le k \le n$) divides the polygon into two, namely a $(k{+}1)$-gon $P_1 = (0, 1, \dots, k)$ and an $(n{-}k{+}3)$-gon $P_2 = (k, k{+}1, \dots, n{+}1, 0)$. The triangulation of the latter is arbitrary, so number of triangulations of P_2 is T_{n-k+1}. The triangulation of the former satisfies the following condition: it has no diagonals from 0. Thus, the number of triangulations of P_1 is equal to the number of triangulations of the k-gon $(1, \dots, k)$, which is equal to T_{k-2}.

Therefore,

$$T_n = T_{n-1} \cdot 1 + \sum_{k=2}^{n} T_{k-2} \cdot T_{n-k+1} = T_{n-1} \cdot T_0 + \sum_{i=1}^{n-1} T_{i-1} \cdot T_{n-i} = \sum_{i=1}^{n} T_{i-1} \cdot T_{n-i}.$$

Alternatively, one can consider the triangle of a triangulation with side $(0, n+1)$, denote the third vertex by $k+2$ (here $2 \le k+2 \le n$, so $0 \le k \le n-2$). It subdivides the polygon into $(k+2)$-gon, a triangle, and a $(n-k+1)$-gon, so the result follows.

1.4. Find a bijection between ballot sequences of length $2n$ and bracketings of $n+1$ variables. *Hint*: assign to every $+1$ in the sequence an opening bracket.

Solution: Assign to every $+1$ in the sequence an opening bracket, and to every -1 a variable. Add another variable at the end. We claim that there is a unique way to add n closing brackets to create a bracketing.

Indeed, we can proceed by induction. The case $n = 1$ is obvious. Now let us go from left to right, find the first two variables $a_i a_{i+1}$ not separated by an opening bracket (note there is always a pair of variables without an opening bracket between them: there are $n+1$ variables, n opening brackets, and the very first symbol is an opening bracket). This means that there is an opening bracket before a_i, so we put a closing bracket after a_{i+1}, and then substitute the expression $(a_i a_{i+1})$ by a new variable a_i', thus reducing the number of variables (and brackets) by one, so we can now use the induction assumption.

The inverse map is obvious: assign $+1$ to every opening bracket, and -1 to every variable except for the last one.

1.5. Given a ballot sequence $\varepsilon_1, \dots, \varepsilon_{2n}$, one can write a sequence of differences $a_i = \varepsilon_{i+1} - \varepsilon_i$, $1 \le i \le 2n-1$. Characterize all such sequences (and thus, get another definition of Catalan numbers).

Solution: Observe first that $a_1 + \dots + a_k = (\varepsilon_2 - \varepsilon_1) + \dots + (\varepsilon_{k+1} - \varepsilon_k) = \varepsilon_{k+1} - \varepsilon_1 = \varepsilon_{k+1} - 1$. Therefore, we can express ε_{k+1} in terms of a_i as $\varepsilon_{k+1} = 1 + \sum_{i=1}^{k} a_i$. We can now reformulate the definition of the ballot sequences in terms of a_i: the three conditions from Definition 1.1 become

(1) $1 + \sum_{i=1}^{k} a_i = \pm 1$, or, equivalently, $\sum_{i=1}^{k} a_i = 0$ or -2 for every positive $k \le 2n-1$;

(2) $1 + \sum_{k=1}^{2n-1} \left(1 + \sum_{i=1}^{k} a_i\right) = 0$, or, equivalently, $\sum_{k=1}^{2n-1} (\sum_{i=1}^{k} a_i) = -2n$;

(3) $1 + \sum_{k=1}^{m} \left(1 + \sum_{i=1}^{k} a_i\right) \ge 0$, or, equivalently, $\sum_{k=1}^{m} (\sum_{i=1}^{k} a_i) \ge -1 - m$ for every positive $m \le 2n-2$.

Once we have a sequence satisfying the three properties above, we can always define $\varepsilon_1 = 1$ and $\varepsilon_{k+1} = 1 + \sum_{i=1}^{k} a_i$ for $k = 1, \dots, 2n-1$.