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Topics in Combinatorics IV, Solutions 11 (Week 11)

11.1. Let P be a cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. A *symmetry* of P is a $g \in O_3(\mathbb{R})$ taking P to itself.

(a) Show that symmetries of P compose a group, denote it by $\operatorname{Sym} P$.

(b) Show that $\operatorname{Sym} P$ acts on the set of faces of P transitively.

(c) Show that $\operatorname{Sym} P$ acts transitively on the set of triples (v, e, f), where v is a vertex of P, e is an edge, f is a face, and $v \in e \subset f$.

(d) An element $g \in O_3(\mathbb{R})$ is *orientation-preserving* if $\det g = 1$. Show that the subgroup of $\operatorname{Sym}^+ P$ of $\operatorname{Sym} P$ consisting of all orientation-preserving symmetries of P is isomorphic to S_4; what does it permute?

(e) Compute the order of $\operatorname{Sym} P$.

Solution:

(a) If $g_1, g_2 \in O_3(\mathbb{R})$ take P to itself, then so do g_i^{-1} and $g_1 g_2$.

(b) A rotation by $\pi/2$ around any coordinate axis belongs to $\operatorname{Sym} P$. Rotations around the z-axis act transitively on vertical faces, and rotations around x-axis take some vertical faces to horizontal ones.

(c) Let (v, e, f) and (v', e', f') be two triples (these are called *flags*). Using rotations around the coordinate axis, we can take v to v', let $g_1 v = v'$, $g_1 \in \operatorname{Sym} P$. Denote $e'' = g_1 e$, it is an edge incident to v'. Now, $\operatorname{Sym} P$ contains a rotation by $2\pi/3$ with respect to the diagonal of P going through v' and $-v'$, powers of this rotation act transitively on all edges containing v'. Thus, there is $g_2 \in \operatorname{Sym} P$ such that $g_2 e'' = e'$ and $g_2 v' = v'$. So, we have already taken v to v' and e to e' by $g_2 g_1 \in \operatorname{Sym} P$. Denote $f'' = g_2 g_1 f$, if $f'' = f'$ then we are done. Otherwise, observe that there are precisely two faces containing e', and they are taken to each other by a reflection in the plane passing through e' and the opposite edge $-e'$. Applying this reflection, we take f'' to f', preserving e' and v'.

(d) Take four main diagonals of P (connecting a vertex with its opposite), $\operatorname{Sym} P$ clearly permutes them. As it was shown above, using rotations only any two diagonals can be taken to any two. Rotating, if needed, the cube by π around the normal to the plane containing these two diagonals, we permute the other two diagonals. Therefore, any permutation of 4 diagonals can be done by orientation-preserving elements of $\operatorname{Sym} P$.

Further, let $g \in \operatorname{Sym}^+ P$ preserve all four diagonals. Choose two diagonals, then g preserves the plane Π containing them. The restriction of g onto Π is a rotation preserving both diagonals, so it is either trivial or a rotation by π. Since $g \in SO_3(\mathbb{R})$, it follows that g either is trivial or is a rotation around the normal to Π (as every element of $SO_3(\mathbb{R})$ has an eigenvalue 1). In the latter case g permuted two other diagonals, so we conclude g is trivial. Thus, $\operatorname{Sym}^+ P$ provides all permutations of diagonals and does not fix any configuration, so it is isomorphic to S_4.

(e) One way to prove this is to observe that the index of $\mathrm{Sym}^+ P$ in $\mathrm{Sym}\, P$ is clearly equal to 2 (by the definition of left cosets), so $|\mathrm{Sym}\, P| = 2|S_4| = 48$.

Alternatively, the number of flags is $8 \cdot 3 \cdot 2 = 48$ (where 8 is the number of vertices, 3 is the number of edges incident to a given vertex, and 2 is the number of faces containing a given edge), the group $\mathrm{Sym}\, P$ acts on the set of flags with a single orbit and trivial stabilizer of every flag. Therefore, $|\mathrm{Sym}\, P| = 48$.

11.2. (a) Show that $\mathrm{Sym}\, P$ is generated by reflections. How many of them do you need to generate $\mathrm{Sym}\, P$?

(b) Show that $\mathrm{Sym}\, P$ cannot be generated by two reflections.

Solution:

(a) We have shown above that to take any flag to any flag we need two types of rotations (with respect to coordinate axes and with respect to normals to planes passing through two diagonals), and reflections with respect to planes passing through two diagonals (or, equivalently, passing through a pair of opposite edges). Now, every rotation of two types above is a product of two reflections. Indeed, $\mathrm{Sym}\, P$ contains the reflections in the following mirrors:

- passing through a coordinate axis and a pair of opposite edges (there are six of these, their equations are $x = \pm z$, $x = \pm y$ and $y = \pm z$);
- passing through two coordinate axes (there are three of them, their equations are $x = 0$, $y = 0$ and $z = 0$).

The rotation around, say, x-axis is a product of the reflections in the plane $z = 0$ and in the plane $x = y$, and similarly for rotations around other coordinate axes; the rotation around the normal to the plane $x = y$ is a product of the reflections in the plane $z = 0$ and in the plane $x = -y$, and similarly for other rotations.

(b) If we assume that $\mathrm{Sym}\, P$ is generated by two reflections r_α and r_β, then the group would leave the space $\{\alpha, \beta\}^\perp$ of positive dimension invariant. However, $\mathrm{Sym}\, P$ is clearly irreducible.

Let v be a vertex of P, $e \ni v$ be an edge of P, and $f \supset e$ be a face of P. Let $p_1 = v$, denote by p_2 the center of e, by p_3 the center of f, and by O the center of P (i.e., the origin of $\mathbb{R}^3$). Let C be the cone over triangle $p_1p_2p_3$ with apex O.

11.3. ($\star$) Show that three reflections in the walls of C generate $\mathrm{Sym}\, P$. Write down the relations among these generators (i.e., give a presentation of $\mathrm{Sym}\, P$ by generators and relations, where generators are the three reflections above).

Solution: Choose $v = (1,1,1)$, $e = P \cap l$ for $l = \{(x,y,z) \mid x = y = 1\}$, $f = P \cap \Pi$ for $\Pi = \{(x,y,z) \mid x = 1\}$. Then $p_1 = v = (1,1,1)$, $p_2 = (1,1,0)$, $p_3 = (1,0,0)$. Then the planes of reflections are $0p_2p_3 = \{z = 0\}$, $0p_1p_3 = \{y = z\}$, $0p_1p_2 = \{x = y\}$. Denote reflections in these planes by s_1, s_2 and s_3 respectively.

The subgroup generated by s_2 and s_3 permutes coordinate axes, so conjugating the generators by the elements of this subgroup we get reflections in planes $x = 0$, $y = 0$ and $x = z$ (cf. Lemma 7.6). Further, $s_1s_2s_1$ is the reflection in the plane $y = -z$ (check this!). Again, conjugating it by s_3 and $s_3s_2s_3$ we get reflections in the planes $x = -y$ and $x = -z$.

Therefore, we got all 9 reflections contained in $\mathrm{Sym}\, P$, which generate the group (see Q11.2).

Of course, one could apply Theorem 7.7 instead (but then one needs to argue why C is a chamber).

We are left to find the relations, i.e. the orders of $s_i s_j$. Reflections s_1 and s_2 generate a dihedral group preserving f, so $(s_1 s_2)^4 = \mathrm{id}$. Reflections s_2 and s_3 generate a group permuting the coordinate axes, so $(s_2 s_3)^3 = \mathrm{id}$. Finally, the reflections s_1 and s_3 commute (as the planes $0p_1p_2$ and $0p_2p_3$ are orthogonal), so

$$\operatorname{Sym} P = \langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^4, (s_2 s_3)^3, (s_1 s_3)^2 \rangle.$$

Let G be a group acting on a set X. Recall that the *stabilizer* $\operatorname{Stab}_G(x)$ of $x \in X$ in G is the set of elements of G fixing x, i.e. $\operatorname{Stab}_G(x) = \{g \in G \mid gx = x\}$. For a set $U \subset X$ the stabilizer $\operatorname{Stab}_G(U)$ is defined as the intersection of stabilizers of all points of U.

11.4. Show that for every point $p \in \mathbb{R}^n$ the stabilizer $\operatorname{Stab}_{\operatorname{Sym} P}(p)$ is generated by all reflections $r \in \operatorname{Sym} P$ such that $rp = p$.

Solution: For general solution see Exercise 12.1.

For $\operatorname{Sym} P$-specific proof one can take a look at the edges and faces of the chambers and see that the statement is obviously true. For points that do not lie at the boundary of chambers the stabilizers are trivial by Theorem 7.7.