Durham University
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Epiphany 2025

Topics in Combinatorics IV, Homework 11 (Week 11)

Due date for starred problems: **Monday, February 03, 12pm**.

Let P be a cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. A *symmetry* of P is a $g \in O_3(\mathbb{R})$ taking P to itself.

11.1. (a) Show that symmetries of P compose a group, denote it by $\operatorname{Sym} P$.

(b) Show that $\operatorname{Sym} P$ acts on the set of faces of P transitively.

(c) Show that $\operatorname{Sym} P$ acts transitively on the set of triples (v, e, f), where v is a vertex of P, e is an edge, f is a face, and $v \in e \subset f$.

(d) An element $g \in O_3(\mathbb{R})$ is *orientation-preserving* if $\det g = 1$. Show that the subgroup $\operatorname{Sym}^+ P$ of $\operatorname{Sym} P$ consisting of all orientation-preserving symmetries of P is isomorphic to S_4; what does it permute?

(e) Compute the order of $\operatorname{Sym} P$.

11.2. (a) Show that $\operatorname{Sym} P$ is generated by reflections. How many of them do you need to generate $\operatorname{Sym} P$?

(b) Show that $\operatorname{Sym} P$ cannot be generated by two reflections.

Let v be a vertex of P, $e \ni v$ be an edge of P, and $f \supset e$ be a face of P. Let $p_1 = v$, denote by p_2 the center of e, by p_3 the center of f, and by O the center of P (i.e., the origin of $\mathbb{R}^3$). Let C be the cone over triangle $p_1 p_2 p_3$ with apex O.

11.3. ($\star$) Show that three reflections in the walls of C generate $\operatorname{Sym} P$. Write down the relations among these generators (i.e., give a presentation of $\operatorname{Sym} P$ by generators and relations, where generators are the three reflections above).

Let G be a group acting on a set X. Recall that the *stabilizer* $\operatorname{Stab}_G(x)$ of $x \in X$ in G is the set of elements of G fixing x, i.e. $\operatorname{Stab}_G(x) = \{g \in G \mid gx = x\}$. For a set $U \subset X$ the stabilizer $\operatorname{Stab}_G(U)$ is defined as the intersection of stabilizers of all points of U.

11.4. Show that for every point $p \in \mathbb{R}^n$ the stabilizer $\operatorname{Stab}_{\operatorname{Sym} P}(p)$ is generated by all reflections $r \in \operatorname{Sym} P$ such that $rp = p$.