Durham University
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Topics in Combinatorics IV, Solutions 12 (Week 12)

12.1. Let G be a finite reflection group in $\mathbb{R}^n$. Recall that the *stabilizer* $\operatorname{Stab}_G(p)$ of $p \in \mathbb{R}^n$ in G is the set of elements of G fixing p, i.e. $\operatorname{Stab}_G(p) = \{g \in G \mid gp = p\}$. G is *irreducible* if it has no invariant subspaces (and *reducible* otherwise).

(a) Let p belong to the intersection of two closed chambers of G only (i.e., p belongs to precisely one mirror $\alpha^\perp$). Show that $\operatorname{Stab}_G(p)$ has order 2 (and is generated by r_α).

(b) Let $p \in \mathbb{R}^n$ belong to at least one mirror of G, $p \neq 0$, and let Γ be the group generated by reflections of G fixing p. Show that Γ is a reducible finite reflection group.

(c) Show that every chamber of Γ is a union of chambers of G.

(d) Show that $\operatorname{Stab}_G(p)$ takes any chamber of Γ to another chamber of Γ (i.e., every $g \in \operatorname{Stab}_G(p)$ permutes chambers of Γ).

(e) Show that Γ acts transitively on all chambers C of G such that $p \in \overline{C}$.

(f) Show that $\operatorname{Stab}_G(p) = \Gamma$, i.e. the stabilizer of $p \in \mathbb{R}^n$ is generated by all reflections $r \in G$ such that $rp = p$.

Solution:

(a) Let C_1 and C_2 be chambers such that $p \in \overline{C}_1 \cap \overline{C}_2$. Then for every $g \in \operatorname{Stab}_G(p)$, $g(C_1) = C_1$ or $g(C_1) = C_2$. In other words, $g = e$ or $g = r_\alpha$, as required.

(b) This holds by definition: Γ is generated by reflections and is finite as a subgroup of finite group. It is reducible as it preserves the span of p.

(c) As Γ is a subgroup of G, every chamber of Γ is bounded by mirrors of G. Thus, every chamber is composed of several chambers of G (the number is the index $[G : \Gamma] = |G|/|\Gamma|$).

(d) The group $\operatorname{Stab}_G(p)$ takes mirrors containing p to mirrors containing p. Therefore, $\operatorname{Stab}_G(p)$ permutes the mirrors of Γ, and thus preserves the chamber structure of Γ.

(e) Consider a small neighborhood N of p not intersecting mirrors of G not containing p, N is tessellated by chambers of Γ on which Γ acts transitively (see Theorem 7.7). At the same time, as N does not intersect other mirrors of G, its intersection with any chamber of Γ is contained in one chamber of G only. Thus, every chamber of Γ contains a unique chamber of G whose closure contains p, and Γ acts on these chambers transitively.

(f) Take any $g \in \operatorname{Stab}_G(p)$, choose any chamber C of G such that $p \in \overline{C}$. Then $p \in \overline{gC}$ as well. As it was proved above, there exists an element of Γ taking C to gC, and thus $g \in \Gamma$.

12.2. $(\star)$

(a) Let $G = I_2(3)(= S_3) = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1s_2)^3\rangle$. Show that all reflections of G are conjugated to each other in G.

(b) For $G = I_2(m) = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1s_2)^m\rangle$, is it true that all reflections in G are conjugated to each other?

(c) Same question for $G = \operatorname{Sym} P$, where P is a 3-dimensional cube (see Exercise 11.3).

Solution:

(a) We know from the proof of Theorem 7.7 that every reflection is conjugated to one of s_i. Also, s_1 is conjugated to s_2 as $(s_1s_2)^3 = e$ is equivalent to $(s_1s_2)s_1(s_2s_1) = s_2$.

(b) This is true for m odd (the proof is as above) and false for m even. Namely, for m even, the group contains an even number of mirrors, and they can be colored in two (alternating) colors representing two conjugacy classes. It is easy to see that both s_1 and s_2 preserve the coloring, so the whole group does.

If we consider the dihedral group as the group of symmetries of a regular m-gon, then for m even one conjugacy class contains reflections whose mirrors pass through a pair of opposite vertices, and the other contains reflections whose mirrors pass through a pair of midpoints of opposite edges.

(c) The statement is false in this case as well. The reflections in coordinate planes are not conjugated to reflections in the planes passing through two opposite edges. Again, it is sufficient to check that two classes are not mixed by the three generating reflections only.

12.3. Show that S_{n+1} has a presentation

$$S_{n+1} = \langle s_1, \dots, s_n \mid s_i^2, (s_is_j)^3 \text{ for } |i-j| = 1, (s_is_j)^2 \text{ for } |i-j| > 1\rangle$$

Solution:

As we have seen at lectures, S_{n+1} acts on the hyperplane $\sum x_i = 0$ in $\mathbb{R}^{n+1}$ by permutations of coordinate axes, and the transpositions (ij) are reflections in planes $x_i = x_j$. The group S_{n+1} can be generated by n transpositions $(i\ i{+}1)$, the corresponding mirrors form a polyhedral cone C defined by $x_i - x_{i+1} < 0$ for $i = 1, \dots, n$, or $x_1 < x_2 < x_3 < \dots < x_{n+1}$. The angle between planes $x_i = x_{i+1}$ and $x_{i+1} = x_{i+2}$ is $\pi/3$ (just compute the angle between the normals $e_i - e_{i+1}$ and $e_{i+1} - e_{i+2}$, where $\{e_i\}$ is the standard orthonormal basis, which is $2\pi/3$), and the angle between $x_i = x_{i+1}$ and $x_j = x_{j+1}$ is right if $|i - j| > 1$.

There are many ways to show that C is a chamber. The easiest way is to observe that no mirror $x_i = x_j$ intersects C – indeed, if $i < j$ then $x_i < x_j$ in C. Another way is to observe that S_{n+1} acts on copies of C obtained by arranging x_i's in different order, and such cones do not overlap. As there are $(n+1)!$ such cones, C must be a fundamental domain for the action, and thus it is a fundamental chamber.

12.4. (a) Let s_1, s_2, s_3 be the three reflections generating the symmetry group of a 3-dimensional cube constructed in Exercise 11.3. Consider all six elements of $\operatorname{Sym} P$ of type $s_i s_j s_k$ for all i, j, k distinct. Show that all six elements are conjugated to each other in $\operatorname{Sym} P$.

(b) Compute the order of these six elements.

Solution:

(a) As s_1 and s_3 commute, there are four distinct elements only to consider: $s_1s_2s_3$, $s_3s_2s_1$, $s_2s_3s_1$ and $s_3s_1s_2$. They all are "cyclically" conjugated: $s_2s_3s_1 = s_2(s_3s_1s_2)s_2 = s_1(s_1s_2s_3)s_1$, while $s_3s_2s_1 = s_1(s_1s_3s_2)s_1 = s_1(s_3s_1s_2)s_1$.

(b) First, the order of all elements is the same (as they are conjugated), and it must be even (as all relations in $\operatorname{Sym} P$ have even length), so we need to look at the order of the square, so consider, say, $(s_1s_2s_3)^2$.

One (purely combinatorial) way to proceed is to use relations (see Exercise 11.3) to simplify the product $(s_1s_2s_3)^2$. We can write

$$(s_1s_2s_3)^2 = s_1s_2s_3s_1s_2s_3 = s_1s_2s_1s_3s_2s_3 = s_1s_2s_1s_2s_3s_2 = s_2s_1s_2s_1s_3s_2 =$$
$$= s_2(s_1s_2s_1s_3)s_2 = s_2(s_1s_2s_3s_1)s_2 = (s_2s_1)s_2s_3(s_1s_2) = (s_2s_1)s_2s_3(s_2s_1)^{-1}$$

(where the blue subword is the one transformed at each step using the relations), so $(s_1s_2s_3)^2$ is conjugated to s_2s_3. The order of s_2s_3 is 3 by the relations, and thus the order of $s_1s_2s_3$ is equal to 6.

Alternatively, one could observe how the element $s_1s_2s_3$ acts on $\mathbb{R}^3$. For this, one could look at the orbits of vertices of P. It takes $v = (1, 1, 1)$ to $(1, 1, -1)$, which is taken to $(1, -1, -1)$, which, in its turn, is taken to $(-1, -1, -1)$. Thus, $(s_1s_2s_3)^3$ takes v to $-v$, and therefore the order of $s_1s_2s_3$ is divisible by 6.

The orbit of v contains six vertices, all six return to their places after application of $(s_1s_2s_3)^6$. The remaining ones are opposite vertices $(1, -1, 1)$ and $(-1, 1, -1)$, $s_1s_2s_3$ either permutes them or leaves intact (actually, it does permute them, but this is not important) – in both cases $(s_1s_2s_3)^6$ leaves them intact. Therefore, $(s_1s_2s_3)^6$ fixes all vertices of P, so it must be an identity.

Alternatively, there is the most straightforward way to proceed: write down the matrices for s_i in the standard basis, compute the eigenvalues of the product $s_1s_2s_3$, and thus compute the order (and much more!).

This is not as difficult as it may look like: the matrices for s_i are

$$s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad s_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so

$$s_1s_2s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},$$

its characteristic polynomial is $-x^3 - 1$, which has roots -1 and two primitive cubic roots of -1, so the result follows.