Durham University
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Epiphany 2025

Topics in Combinatorics IV, Homework 12 (Week 12)

Due date for starred problems: **Monday, February 03, 12pm**.

12.1. Let G be a finite reflection group in $\mathbb{R}^n$. Recall that the *stabilizer* $\mathrm{Stab}_G(p)$ of $p \in \mathbb{R}^n$ in G is the set of elements of G fixing p, i.e. $\mathrm{Stab}_G(p) = \{g \in G \mid gp = p\}$. G is *irreducible* if it has no invariant subspaces (and *reducible* otherwise).

(a) Let p belong to the intersection of two closed chambers of G only (i.e., p belongs to precisely one mirror $\alpha^\perp$). Show that $\mathrm{Stab}_G(p)$ has order 2 (and is generated by r_α).

(b) Let $p \in \mathbb{R}^n$ belong to at least one mirror of G, $p \neq 0$, and let Γ be the group generated by reflections of G fixing p. Show that Γ is a reducible finite reflection group.

(c) Show that every chamber of Γ is a union of chambers of G.

(d) Show that $\mathrm{Stab}_G(p)$ takes any chamber of Γ to another chamber of Γ (i.e., every $g \in \mathrm{Stab}_G(p)$ permutes chambers of Γ).

(e) Show that Γ acts transitively on all chambers C of G such that $p \in \overline{C}$.

(f) Show that $\mathrm{Stab}_G(p) = \Gamma$, i.e. the stabilizer of $p \in \mathbb{R}^n$ is generated by all reflections $r \in G$ such that $rp = p$.

12.2. ($\star$)

(a) Let $G = I_2(3)(= S_3) = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1s_2)^3 \rangle$. Show that all reflections of G are conjugated to each other in G.

(b) For $G = I_2(m) = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1s_2)^m \rangle$, is it true that all reflections in G are conjugated to each other?

(c) Same question for $G = \mathrm{Sym}\, P$, where P is a 3-dimensional cube (see Exercise 11.3).

12.3. Show that S_{n+1} has a presentation

$$S_{n+1} = \langle s_1, \dots, s_n \mid s_i^2, (s_is_j)^3 \text{ for } |i-j| = 1, (s_is_j)^2 \text{ for } |i-j| > 1 \rangle$$

12.4. (a) Let s_1, s_2, s_3 be the three reflections generating the symmetry group of a 3-dimensional cube constructed in Exercise 11.3. Consider all six elements of $\mathrm{Sym}\, P$ of type $s_is_js_k$ for all i, j, k distinct. Show that all six elements are conjugated to each other in $\mathrm{Sym}\, P$.

(b) Compute the order of these six elements.