## Topics in Combinatorics IV, Solutions 13 (Week 13)

13.1. (a) Let $G$ be a simply-laced irreducible finite reflection group (i.e., all $m_{i j}$ are equal to 2 or 3 ), or one of $H_{3}$ and $H_{4}$. Show that all reflections of $G$ are conjugated in $G$.
(b) Let $G$ be a finite reflection group, and let $r_{1}, r_{2} \in G$ be two reflections. Show that the dihedral subgroup generated by $r_{1}$ and $r_{2}$ is conjugated in $G$ to a subgroup generated by some simple reflections $s_{i}$ and $s_{j}$.

## Solution:

(a) Let $s_{1}, \ldots s_{n}$ be simple reflections. If $s_{i}$ and $s_{j}$ do not commute, then $\left(s_{i} s_{j}\right)^{m_{i j}}=e$ is equivalent to $s_{i}=s_{j} s_{i} s_{j} \ldots s_{i} s_{j}$, where the number of factors in the RHS of the equality is equal to $2 m_{i j}-1$. Since $m_{i j}$ is odd, we get

$$
s_{i}=\underbrace{\left(s_{j} s_{i} \ldots s_{i}\right)}_{m_{i j}-1} s_{j} \underbrace{\left(s_{i} \ldots s_{j}\right)}_{m_{i j}-1}=\left(s_{j} s_{i}\right)^{\frac{m_{i j}-1}{2}} s_{j}\left(s_{i} s_{j}\right)^{\frac{m_{i j}-1}{2}}=\left(s_{j} s_{i}\right)^{\frac{m_{i j}-1}{2}} s_{j}\left(\left(s_{j} s_{i}\right)^{\frac{m_{i j}-1}{2}}\right)^{-1}
$$

Since $G$ is irreducible, the Coxeter diagram is connected. Thus, there is a path between any two vertices, and, as we have proved above, any two simple reflections whose vertices are joined by an edge are conjugated. Therefore, all simple reflections are conjugated.
Finally, every reflection in $G$ is conjugated to one of the simple reflections (see the proof of Theorem 7.7), so all reflections are conjugated.
(b) Take the subgroup generated by $r_{1}$ and $r_{2}$, it has order $2 m$. There exits a reflection $r$ in the subgroup such that the angle between mirrors of $r_{1}$ and $r$ is equal to $\pi / m$. Then these two mirrors bound one chamber $C$. There exists $g \in G$ such that $C=g C_{0}$, where $C_{0}$ is the chamber bounded by mirrors of simple reflections $s_{i}$. Conjugating $r$ and $r_{1}$ by $g^{-1}$, we obtain some $s_{i}$ and $s_{j}$.
13.2. (a) Let $G$ be a Coxeter group defined by $G=\left\langle t_{1}, t_{2}, t_{3} \mid t_{i}^{2},\left(t_{1} t_{2}\right)^{2},\left(t_{1} t_{3}\right)^{2},\left(t_{2} t_{3}\right)^{3}\right\rangle$. Show that $G$ is isomorphic to $I_{2}(6)$.
(b) Show that for any odd $k$ a Coxeter group $G=\left\langle t_{1}, t_{2}, t_{3}, \mid t_{i}^{2},\left(t_{1} t_{2}\right)^{2},\left(t_{1} t_{3}\right)^{2},\left(t_{2} t_{3}\right)^{k}\right\rangle$ is isomorphic to $I_{2}(2 k)$.

## Solution:

(a) There is a unique cyclic subgroup of order 6 in each of the groups: in $I_{2}(6)$ it is generated by $s_{1} s_{2}$, and in $G$ by $t_{1} t_{2} t_{3}$; the center of $I_{2}(6)$ is generated by the rotation by $\pi$, i.e. $\left(s_{1} s_{2}\right)^{3}$, while the center of $G$ is generated by $t_{1}$. Thus, it is natural to look for a homomorphism which takes $t_{1} \mapsto\left(s_{1} s_{2}\right)^{3}, t_{1} t_{2} t_{3} \mapsto s_{1} s_{2}$. This leaves us with $t_{2} t_{3} \mapsto\left(s_{2} s_{1}\right)^{2}$. So, we can try a map

$$
\varphi: t_{1} \mapsto\left(s_{1} s_{2}\right)^{3}, \quad t_{2} \mapsto s_{2}, \quad t_{3} \mapsto s_{1} s_{2} s_{1}
$$

We need to check that this is a homomorphism. Indeed,

$$
\begin{aligned}
\varphi\left(\left(t_{1}\right)^{2}\right) & =\left(s_{1} s_{2}\right)^{6}=e \\
\varphi\left(\left(t_{2}\right)^{2}\right) & =s_{2}^{2}=e \\
\varphi\left(\left(t_{3}\right)^{2}\right) & =s_{1} s_{2} s_{1} s_{1} s_{2} s_{1}=e \\
\varphi\left(\left(t_{1} t_{2}\right)^{2}\right) & =\left(s_{1} s_{2} s_{1} s_{2} s_{1}\right)^{2}=e \\
\varphi\left(\left(t_{1} t_{3}\right)^{2}\right) & =\left(s_{1} s_{2} s_{1} s_{2} s_{1} s_{2} s_{1} s_{2} s_{1}\right)^{2}=e \\
\varphi\left(\left(t_{2} t_{3}\right)^{3}\right) & =\left(\left(s_{2} s_{1}\right)^{2}\right)^{3}=e
\end{aligned}
$$

The map is clearly surjective as $\varphi\left(t_{2}\right)=s_{2}$ and $\varphi\left(t_{1} t_{2} t_{3} t_{2}\right)=s_{1}$, so it is injective as well (as the order of both groups is 12), and thus we got an isomorphism of groups.
(b) The solution is similar to (a): take

$$
\varphi: t_{1} \mapsto\left(s_{1} s_{2}\right)^{k}, \quad t_{2} \mapsto s_{2}, \quad t_{3} \mapsto\left(s_{1} s_{2}\right)^{k-2} s_{1}
$$

The computations are identical to (a), except for $\varphi\left(\left(t_{2} t_{3}\right)^{k}\right)=\left(\left(s_{2} s_{1}\right)^{k-1}\right)^{k}=e$ as $k-1$ is even.
13.3. $(\star)$ Let $\left\{e_{1}, e_{2}, e_{3}\right\}$ be a standard orthonormal basis of $\mathbb{R}^{3}$. Let $G$ be the group generated by reflections in all vectors of the form $e_{i} \pm e_{j}, i<j$.
(a) Show that $G$ does not contain any other reflections.
(b) Find a triple of reflections $\left\{s_{1}, s_{2}, s_{3}\right\}$ of $G$ such that the angles formed by the outward normals $\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ to the mirrors of $s_{i}$ are all non-acute (i.e., $\pi / 2$ or larger).
Note: there are many such triples.
(c) Write down the Gram matrix of $\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$, i.e. the matrix $\left(a_{i j}\right)$ with $a_{i j}=\left(\alpha_{i}, \alpha_{j}\right)$. Draw the corresponding Coxeter diagram.
(d) Write down the presentation of $G$ as a Coxeter group. Find the order of $G$.

## Solution:

(a) By Theorem 7.7, all reflections in the group are of the form $r_{\alpha}$, where $\alpha=g\left(e_{i} \pm e_{j}\right)$ for some $g \in G$. However, the generators of $G$ preserve the set $\left\{ \pm e_{i} \pm e_{j}\right\}$, and thus the whole group also preserves this set. Therefore, there are no other reflections.
(b) One can take $\alpha_{1}=e_{2}-e_{3}, \alpha_{2}=e_{1}-e_{2}, \alpha_{3}=e_{2}+e_{3}$. Then $\left(\alpha_{1}, \alpha_{2}\right)=\left(\alpha_{2}, \alpha_{3}\right)=-1$, and $\left(\alpha_{1}, \alpha_{3}\right)=0$.
(c) The Gram matrix is

$$
\left(\begin{array}{ccc}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{array}\right)
$$

the Coxeter diagram is $A_{3}$.
(d) $G=\left\langle s_{1}, s_{2}, s_{3} \mid s_{i}^{2},\left(s_{1} s_{2}\right)^{3},\left(s_{1} s_{3}\right)^{2},\left(s_{2} s_{3}\right)^{3}\right\rangle$. In fact $G$ is the symmetric group $S_{4}$, so its order is 24 .

