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Topics in Combinatorics IV, Solutions 13 (Week 13)

13.1. (a) Let G be a simply-laced irreducible finite reflection group (i.e., all m_{ij} are equal to 2 or 3), or one of H_3 and H_4. Show that all reflections of G are conjugated in G.

(b) Let G be a finite reflection group, and let $r_1, r_2 \in G$ be two reflections. Show that the dihedral subgroup generated by r_1 and r_2 is conjugated in G to a subgroup generated by some simple reflections s_i and s_j.

Solution:

(a) Let $s_1, \dots s_n$ be simple reflections. If s_i and s_j do not commute, then $(s_i s_j)^{m_{ij}} = e$ is equivalent to $s_i = s_j s_i s_j \dots s_i s_j$, where the number of factors in the RHS of the equality is equal to $2m_{ij} - 1$. Since m_{ij} is odd, we get

$$s_i = \underbrace{(s_j s_i \dots s_i)}_{m_{ij}-1} s_j \underbrace{(s_i \dots s_j)}_{m_{ij}-1} = (s_j s_i)^{\frac{m_{ij}-1}{2}} s_j (s_i s_j)^{\frac{m_{ij}-1}{2}} = (s_j s_i)^{\frac{m_{ij}-1}{2}} s_j \left((s_j s_i)^{\frac{m_{ij}-1}{2}} \right)^{-1}$$

Since G is irreducible, the Coxeter diagram is connected. Thus, there is a path between any two vertices, and, as we have proved above, any two simple reflections whose vertices are joined by an edge are conjugated. Therefore, all simple reflections are conjugated.

Finally, every reflection in G is conjugated to one of the simple reflections (see the proof of Theorem 7.7), so all reflections are conjugated.

(b) Take the subgroup generated by r_1 and r_2, it has order $2m$. There exits a reflection r in the subgroup such that the angle between mirrors of r_1 and r is equal to π/m. Then these two mirrors bound one chamber C. There exists $g \in G$ such that $C = gC_0$, where C_0 is the chamber bounded by mirrors of simple reflections s_i. Conjugating r and r_1 by g^{-1}, we obtain some s_i and s_j.

13.2. (a) Let G be a Coxeter group defined by $G = \langle t_1, t_2, t_3 \mid t_i^2, (t_1t_2)^2, (t_1t_3)^2, (t_2t_3)^3 \rangle$. Show that G is isomorphic to $I_2(6)$.

(b) Show that for any odd k a Coxeter group $G = \langle t_1, t_2, t_3, \mid t_i^2, (t_1t_2)^2, (t_1t_3)^2, (t_2t_3)^k \rangle$ is isomorphic to $I_2(2k)$.

Solution:

(a) There is a unique cyclic subgroup of order 6 in each of the groups: in $I_2(6)$ it is generated by s_1s_2, and in G by $t_1t_2t_3$; the center of $I_2(6)$ is generated by the rotation by π, i.e. $(s_1s_2)^3$, while the center of G is generated by t_1. Thus, it is natural to look for a homomorphism which takes $t_1 \mapsto (s_1s_2)^3$, $t_1t_2t_3 \mapsto s_1s_2$. This leaves us with $t_2t_3 \mapsto (s_2s_1)^2$. So, we can try a map

$$\varphi : t_1 \mapsto (s_1s_2)^3, \quad t_2 \mapsto s_2, \quad t_3 \mapsto s_1s_2s_1.$$

We need to check that this is a homomorphism. Indeed,

$$\begin{aligned}
\varphi((t_1)^2) &= (s_1s_2)^6 = e\\
\varphi((t_2)^2) &= s_2^2 = e\\
\varphi((t_3)^2) &= s_1s_2s_1s_1s_2s_1 = e\\
\varphi((t_1t_2)^2) &= (s_1s_2s_1s_2s_1)^2 = e\\
\varphi((t_1t_3)^2) &= (s_1s_2s_1s_2s_1s_2s_1s_2s_1)^2 = e\\
\varphi((t_2t_3)^3) &= ((s_2s_1)^2)^3 = e
\end{aligned}$$

The map is clearly surjective as $\varphi(t_2) = s_2$ and $\varphi(t_1t_2t_3t_2) = s_1$, so it is injective as well (as the order of both groups is 12), and thus we got an isomorphism of groups.

(b) The solution is similar to (a): take

$$\varphi : t_1 \mapsto (s_1s_2)^k, \quad t_2 \mapsto s_2, \quad t_3 \mapsto (s_1s_2)^{k-2}s_1.$$

The computations are identical to (a), except for $\varphi((t_2t_3)^k) = ((s_2s_1)^{k-1})^k = e$ as $k-1$ is even.

13.3. ($\star$) Let $\{e_1, e_2, e_3\}$ be a standard orthonormal basis of $\mathbb{R}^3$. Let G be the group generated by reflections in all vectors of the form $e_i \pm e_j$, $i < j$.

(a) Show that G does not contain any other reflections.

(b) Find a triple of reflections $\{s_1, s_2, s_3\}$ of G such that the angles formed by the outward normals $\{\alpha_1, \alpha_2, \alpha_3\}$ to the mirrors of s_i are all non-acute (i.e., $\pi/2$ or larger).
Note: there are *many* such triples.

(c) Write down the *Gram matrix* of $\{\alpha_1, \alpha_2, \alpha_3\}$, i.e. the matrix (a_{ij}) with $a_{ij} = (\alpha_i, \alpha_j)$. Draw the corresponding Coxeter diagram.

(d) Write down the presentation of G as a Coxeter group. Find the order of G.

Solution:

(a) By Theorem 7.7, all reflections in the group are of the form r_α, where $\alpha = g(e_i \pm e_j)$ for some $g \in G$. However, the generators of G preserve the set $\{\pm e_i \pm e_j\}$, and thus the whole group also preserves this set. Therefore, there are no other reflections.

(b) One can take $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_1 - e_2$, $\alpha_3 = e_2 + e_3$. Then $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1$, and $(\alpha_1, \alpha_3) = 0$.

(c) The Gram matrix is

$$\begin{pmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2 \end{pmatrix},$$

the Coxeter diagram is A_3.

(d) $G = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_1s_3)^2, (s_2s_3)^3 \rangle$. In fact G is the symmetric group S_4, so its order is 24.