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Topics in Combinatorics IV, Solutions 14 (Week 14)

14.1. ($\star$) Let (G, S) be a Coxeter system.

(a) Let u, v be words, and let $r(w)$ denote the R-sequence of a word w. Show that $r(uv) = (r(u), ur(v)u^{-1})$.

(b) Let $w = s_1 \dots s_k$ be a word. Show that there exists an increasing sequence of indices $1 \le i_1 < i_2 < \dots < i_m \le k$, such that $s_{i_1} \dots s_{i_m}$ is reduced and equivalent to w in G.

(c) Show that the order of any finite Coxeter group is even.

Solution:

(a) Let $u = s_1 \dots s_k$, $v = s_{k+1} \dots s_m$. Then

$$r(uv) = (s_1, s_1s_2s_1, \dots, (s_1 \dots s_{k-1})s_k(s_{k-1} \dots s_1),$$
$$(s_1 \dots s_k)s_{k+1}(s_k \dots s_1), (s_1 \dots s_{k+1})s_{k+2}(s_{k+1} \dots s_1), \dots, (s_1 \dots s_{m-1})s_m(s_{m-1} \dots s_1)) =$$
$$= (\underbrace{s_1, s_1s_2s_1, \dots, (s_1 \dots s_{k-1})s_k(s_{k-1} \dots s_1)}_{r(u)},$$
$$\underbrace{(s_1 \dots s_k)s_{k+1}(s_k \dots s_1), (s_1 \dots s_k)s_{k+1}s_{k+2}s_{k+1}(s_k \dots s_1), \dots, (s_1 \dots s_k)s_{k+1} \dots s_m \dots s_{k+1}(s_k \dots s_1)}_{ur(v)u^{-1}})$$

(b) By Deletion Condition, if w is not reduced then $w \sim_G s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ for some $i < j$. Applying Deletion Condition iteratively until the result is reduced, we get the required statement.

(c) All relations in a Coxeter group have even length. Therefore, the set of elements of even length is a normal subgroup of index two.

Alternatively, Lagrange's theorem says that the order of any subgroup divides the order of the group, which implies that the order of any element divides the order of the group. Now, any element of S has order two.

14.2. Let G be any group with a finite generating set S. Assume also that S is symmetric, i.e. for any $s \in S$ the inverse s^{-1} is also contained in S. Let $g, g' \in G$, and let $l(g)$ denote the shortest length of a reduced word representing g.

(a) Show that $|l(g) - l(g')| \leq l(g'g^{-1})$.

(b) Show that $d(g, g') = l(g'g^{-1})$ defines a metric on G.

Solution:

(a) This follows from Lemma 8.3(1): as $g' = (g'g^{-1})g$, we have $l(g') \leq l(g'g^{-1}) + l(g)$, which is equivalent to the required inequality.

(b) We need to verify four conditions: non-negativity, symmetry, $d(g, g') = 0 \Leftrightarrow g = g'$, and the triangle inequality.

Non-negativity holds by definition of the length. Symmetry holds by Lemma 8.3(2):

$$d(g, g') = l(g'g^{-1}) = l((g'g^{-1})^{-1}) = l(gg'^{-1}) = d(g', g)$$

The equality $d(g, g') = 0$ is equivalent to $l(g'g^{-1}) = 0$, which is the same as $g'g^{-1} = e$, so $g = g'$.

Finally, we need to prove that $d(g_1, g_2) \leq d(g_1, g) + d(g, g_2)$. Indeed,

$$d(g_1, g) + d(g, g_2) = l(gg_1^{-1}) + l(g_2g^{-1}) \geq l((g_2g^{-1})(gg_1^{-1})) = l(g_2g_1^{-1}) = d(g_1, g_2)$$

by Lemma 8.3(1).

14.3. Let (G, S) be a Coxeter system. Given $s \in S$, denote by P_s the set of $g \in G$ such that $l(sg) > l(g)$.

(a) Show that $\bigcap_{s \in S} P_s = \{e\}$.

(b) Show that for $s \in S$ and $g \in G$ either $l(sg) > l(g)$ or $l(sg) < l(g)$.

(c) Show that for every $s \in S$ the sets P_s and sP_s do not intersect, and their union is G (i.e., they form a *partition* of G).

(d) Let $s, t \in S$, $g \in G$. Show that if $g \in P_s$ and $gt \notin P_s$, then $sg = gt$.

Solution:

(a) Take any $g \in G$ be an arbitrary non-trivial element, and let $w = s_1 \dots s_k$ be a reduced word representing g. Then $s_1 w = s_2 \dots s_k$, so $l(s_1 g) < l(g)$, and thus $g \notin P_{s_1}$.

(b) The parity of the number of reflections in any expression is an invariant: all relations in a Coxeter group are of even length.

(c) By (b), either $l(sg) > l(g)$ or $l(sg) < l(g)$. In the former case $g \in P_s$. In the latter case, $l(s(sg)) = l(g) > l(sg)$, and thus $gs \in P_s$, so $g \in sP_s$.

(d) If $k = l(g)$, then $l(sg) = k + 1$ and $l(sgt) < l(gt) \leq k$. If $l(sgt) < k$, then $l(sg) = l((sgt)t) < k + 1$, so we see that $l(sgt) = k$, and thus $l(gt) = k + 1$.

Let $s_1 \dots s_k t$ be a reduced expression for gt. By Exchange Condition, $gt \sim_G ss_1 \dots \hat{s}_i \dots s_k t$ or $gt \sim_G ss_1 \dots s_k$. In the former case $sg \sim_G s_1 \dots \hat{s}_i \dots s_k$, so $l(sg) < k$, which contradicts the assumptions. In the latter case $gt = sg$.

14.4. Let G be a group with a finite generating set S consisting of involutions, and let $\{P_s\}_{s \in S}$ be a family of subsets of G satisfying the following properties:

(1) $e \in P_s$ for every $s \in S$;

(2) $P_s \cap sP_s = \emptyset$ for every $s \in S$;

(3) For every $s, t \in S$ and $g \in G$ such that $g \in P_s$ and $gt \notin P_s$, one has $sg = gt$.

Show that $P_s = \{g \in G \mid l(sg) > l(g)\}$, and (G, S) satisfies Exchange Condition (and thus is a Coxeter system).

Solution:

Take any $g \in G$ and $s \in S$, then either $g \in P_s$ or not.

First, assume $g \notin P_s$. Let $w = s_1 \ldots s_k$ be a reduced expression for g. Denote $g_i = s_1 \ldots s_i$. There exists i such that $g_i \notin P_s$, but for all $j < i$ we have $g_j \in P_s$ (for $j = 0$ we denote $g_0 = e \in P_s$ by (1)).

Thus, we have $g_{i-1} \in P_s$, $g_i = g_{i-1}s_i \notin P_s$, and (3) implies $sg_{i-1} = g_{i-1}s_i$, or, equivalently, $g \sim_G ss_1 \ldots \hat{s}_i \ldots s_k$. In particular, $l(sg) < l(g)$.

Next, assume $g \in P_s$. By (2), $g \notin sP_S$, i.e. $sg \notin P_s$. By the previous part, this implies $l(g) = l(s(sg)) < l(sg)$.

Therefore, we have proved that $g \in P_s$ if and only if $l(g) < l(sg)$, and if $l(sg) > l(g)$ (i.e., $g \notin P_s$) then the Exchange Condition holds.