Durham University
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Epiphany 2025

Topics in Combinatorics IV, Homework 14 (Week 14)

Due date for starred problems: **Monday, February 17, 12pm**.

14.1. ($\star$) Let (G, S) be a Coxeter system.

(a) Let u, v be words, and let $r(w)$ denote the R-sequence of a word w. Show that $r(uv) = (r(u), ur(v)u^{-1})$.

(b) Let $w = s_1 \dots s_k$ be a word. Show that there exists an increasing sequence of indices $1 \leq i_1 < i_2 < \dots < i_m \leq k$, such that $s_{i_1} \dots s_{i_m}$ is reduced and equivalent to w in G.

(c) Show that the order of any finite Coxeter group is even.

14.2. Let G be any group with a finite generating set S. Assume also that S is symmetric, i.e. for any $s \in S$ the inverse s^{-1} is also contained in S. Let $g, g' \in G$, and let $l(g)$ denote the shortest length of a reduced word representing g.

(a) Show that $|l(g) - l(g')| \leq l(g'g^{-1})$.

(b) Show that $d(g, g') = l(g'g^{-1})$ defines a metric on G.

14.3. Let (G, S) be a Coxeter system. Given $s \in S$, denote by P_s the set of $g \in G$ such that $l(sg) > l(g)$.

(a) Show that $\bigcap\limits_{s \in S} P_s = \{e\}$.

(b) Show that for $s \in S$ and $g \in G$ either $l(sg) > l(g)$ or $l(sg) < l(g)$.

(c) Show that for every $s \in S$ the sets P_s and sP_s do not intersect, and their union is G (i.e., they form a *partition* of G).

(d) Let $s, t \in S$, $g \in G$. Show that if $g \in P_s$ and $gt \notin P_s$, then $sg = gt$.

14.4. Let G be a group with a finite generating set S consisting of involutions, and let $\{P_s\}_{s \in S}$ be a family of subsets of G satisfying the following properties:

(1) $e \in P_s$ for every $s \in S$;

(2) $P_s \cap sP_s = \emptyset$ for every $s \in S$;

(3) For every $s, t \in S$ and $g \in G$ such that $g \in P_s$ and $gt \notin P_s$, one has $sg = gt$.

Show that $P_s = \{g \in G \mid l(sg) > l(g)\}$, and (G, S) satisfies Exchange Condition (and thus is a Coxeter system).