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Topics in Combinatorics IV, Solutions 15 (Week 15)

15.1. Let $\Gamma = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^3, s_3^7 \rangle$. Show that the subgroup generated by s_1 and s_2 is trivial, although the group $\Gamma' = \langle s_1, s_2 \mid \text{all relations not containing } s_3 \rangle$ is not.

Solution: Since $s_3^2 = s_3^7 = e$, we get $s_3 = e$. Now, $(s_is_3)^3 = e$ becomes $s_i^3 = e$, which gives $s_i = e$ in view of $s_i^2 = e$. At the same time, Γ' is a Coxeter group of type A_2.

15.2. Let (G, S) be a Coxeter system, and let $T \subset S$. Define G_T to be the subgroup of G generated by elements of T (G_T is called a *standard parabolic subgroup* of G).

(a) Let $w = s_1 \dots s_k$ be a word, all $s_i \in T$. Show that for any M-reduction $w \to w_0$ all words obtained during the procedure belong to G_T.

(b) Let $\Gamma = \langle T \mid s_i^2, (s_is_j)^{m_{ij}} \rangle$. Define a homomorphism $\varphi : \Gamma \to G$ by $\varphi(s_i) = s_i$. Show that $\ker \varphi$ is trivial.

(c) Show that (G_T, T) is a Coxeter system.

Solution:

(a) This follows from the definition of M-reduction: removing a subword ss leaves the word in G_T, as well as substituting $(st)^{m_{st}}$ with $(ts)^{m_{st}}$ for $s, t \in T$.

(b) Let w be a word in G_T, and assume $w \to e$. By (a), an M-reduction uses relations involving elements of T only. Applying the same procedure to the same word w in Γ, we see that $w \sim_\Gamma e$. Therefore, the kernel of the homomorphism is trivial.

(c) This immediately follows from (b): the isomorphism φ takes a Coxeter system (Γ, T) to (G_T, T).

15.3. ($\star$) Let (G, S) be a Coxeter system, $s, t \in S$, and $m_{st} = \infty$ (i.e., there is no relation on st). Let w be a reduced word. Show that either $s \notin r(w)$ or $t \notin r(w)$.

Solution:

Suppose there is a reduced word w such that $s, t \in r(w)$. Let $g \in G$ be the corresponding group element. By Theorem 8.9, for every word representing g its R-sequence contains s and t. Then the algorithm used in the proof of Lemma 8.17 (together with Lemma 8.16) shows that $g = sts \dots = tst \dots$, where the number of entries is equal to $l(g)$. Then $(st)^{l(g)} = e$, so we come to a contradiction.

15.4. Let (G, S) be Coxeter system, $r \in R$ and $g \in G$. Show that if $r \in R(g)$ then $l(rg) < l(g)$.

Solution: Let $w = s_1 \dots s_k$ be a reduced expression for g, and the R-sequence of w is $(r_1, \dots, r_k)$. Recall that $r_i = (s_1 \dots s_{i-1})s_i(s_{i-1} \dots s_1)$, and $s_1 \dots s_i = r_i \dots r_1$ for any i.

Assume that $r \in R(g)$, so we have $r = r_j$ for some j. Therefore,

$$rg = r_j g = r_j r_k r_{k-1} \dots r_1 = r_j(r_k \dots r_{j+1} r_j)(r_{j-1} \dots r_1)$$

Observe that

$$\begin{aligned} r_k \dots r_j = (r_k \dots r_j r_{j-1} \dots r_1)(r_1 \dots r_{j-1}) = (r_k \dots r_1)(r_{j-1} \dots r_1)^{-1} = \\ = (s_1 \dots s_k)(s_1 \dots s_{j-1})^{-1} = (s_1 \dots s_k)(s_{j-1} \dots s_1) = (s_1 \dots s_{j-1})s_j \dots s_k(s_{j-1} \dots s_1), \end{aligned}$$

so we can compute

$$\begin{aligned} rg = r_j(r_k \dots r_{j+1} r_j)(r_{j-1} \dots r_1) = r_j(s_1 \dots s_{j-1})s_j \dots s_k(s_{j-1} \dots s_1)(r_{j-1} \dots r_1) = \\ = r_j(s_1 \dots s_{j-1})s_j \dots s_k(s_{j-1} \dots s_1)(s_1 \dots s_{j-1}) = r_j(s_1 \dots s_{j-1})s_j \dots s_k = \\ = (s_1 \dots s_{j-1})s_j(s_{j-1} \dots s_1)(s_1 \dots s_{j-1})s_j \dots s_k = (s_1 \dots s_{j-1})s_j s_j \dots s_k = \\ = s_1 \dots s_{j-1} s_{j+1} \dots s_k = s_1 \dots \hat{s}_j \dots s_k \end{aligned}$$

15.5. $(\star)$ Let (G, S) be Coxeter system such that its Coxeter diagram contains a cycle. Find an element of infinite order in G.

Solution:

Let $1 \dots k$ be a cycle. Consider an element $s_1 \dots s_k$. Then for any $m \in \mathbb{N}$ the element $(s_1 \dots s_k)^m$ is M-reduced: there is no elementary M-operation that can be applied to it. By Theorem 8.19, $(s_1 \dots s_k)^m$ is reduced, so $(s_1 \dots s_k)$ has infinite order.