## Topics in Combinatorics IV, Solutions 15 (Week 15)

15.1. Let $\Gamma=\left\langle s_{1}, s_{2}, s_{3} \mid s_{i}^{2},\left(s_{1} s_{2}\right)^{3},\left(s_{2} s_{3}\right)^{3},\left(s_{1} s_{3}\right)^{3}, s_{3}^{7}\right\rangle$. Show that the subgroup generated by $s_{1}$ and $s_{2}$ is trivial, although the group $\Gamma^{\prime}=\left\langle s_{1}, s_{2}\right|$ all relations not containing $\left.s_{3}\right\rangle$ is not.

Solution: Since $s_{3}^{2}=s_{3}^{7}=e$, we get $s_{3}=e$. Now, $\left(s_{i} s_{3}\right)^{3}=e$ becomes $s_{i}^{3}=e$, which gives $s_{i}=e$ in view of $s_{i}^{2}=e$. At the same time, $\Gamma^{\prime}$ is a Coxeter group of type $A_{2}$.
15.2. Let $(G, S)$ be a Coxeter system, and let $T \subset S$. Define $G_{T}$ to be the subgroup of $G$ generated by elements of $T\left(G_{T}\right.$ is called a standard parabolic subgroup of $\left.G\right)$.
(a) Let $w=s_{1} \ldots s_{k}$ be a word, all $s_{i} \in T$. Show that for any $M$-reduction $w \rightarrow w_{0}$ all words obtained during the procedure belong to $G_{T}$.
(b) Let $\Gamma=\left\langle T \mid s_{i}^{2},\left(s_{i} s_{j}\right)^{m_{i j}}\right\rangle$. Define a homomorphism $\varphi: \Gamma \rightarrow G$ by $\varphi\left(s_{i}\right)=s_{i}$. Show that $\operatorname{ker} \varphi$ is trivial.
(c) Show that $\left(G_{T}, T\right)$ is a Coxeter system.

## Solution:

(a) This follows from the definition of $M$-reduction: removing a subword $s s$ leaves the word in $G_{T}$, as well as substituting $(s t)^{m_{s t}}$ with $(t s)^{m_{s t}}$ for $s, t \in T$.
(b) Let $w$ be a word in $G_{T}$, and assume $w \rightarrow e$. By (a), an $M$-reduction uses relations involving elements of $T$ only. Applying the same procedure to the same word $w$ in $\Gamma$, we see that $w \sim_{\Gamma} e$. Therefore, the kernel of the homomorphism is trivial.
(c) This immediately follows from (b): the isomorphism $\varphi$ takes a Coxeter system ( $\Gamma, T$ ) to $\left(G_{T}, T\right)$.
15.3. ( $\star$ ) Let $(G, S)$ be a Coxeter system, $s, t \in S$, and $m_{s t}=\infty$ (i.e., there is no relation on $s t$ ). Let $w$ be a reduced word. Show that either $s \notin r(w)$ or $t \notin r(w)$.

## Solution:

Suppose there is a reduced word $w$ such that $s, t \in r(w)$. Let $g \in G$ be the corresponding group element. By Theorem 8.9, for every word representing $g$ its $R$-sequence contains $s$ and $t$. Then the algorithm used in the proof of Lemma 8.17 (together with Lemma 8.16) shows that $g=s t s \cdots=$ $t s t \ldots$, where the number of entries is equal to $l(g)$. Then $(s t)^{l(g)}=e$, so we come to a contradiction.
15.4. Let $(G, S)$ be Coxeter system, $r \in R$ and $g \in G$. Show that if $r \in R(g)$ then $l(r g)<l(g)$.

Solution: Let $w=s_{1} \ldots s_{k}$ be a reduced expression for $g$, and the $R$-sequence of $w$ is $\left(r_{1}, \ldots, r_{k}\right)$. Recall that $r_{i}=\left(s_{1} \ldots s_{i-1}\right) s_{i}\left(s_{i-1} \ldots s_{1}\right)$, and $s_{1} \ldots s_{i}=r_{i} \ldots r_{1}$ for any $i$.
Assume that $r \in R(g)$, so we have $r=r_{j}$ for some $j$. Therefore,

$$
r g=r_{j} g=r_{j} r_{k} r_{k-1} \ldots r_{1}=r_{j}\left(r_{k} \ldots r_{j+1} r_{j}\right)\left(r_{j-1} \ldots r_{1}\right)
$$

Observe that

$$
\begin{aligned}
r_{k} \ldots r_{j}= & \left(r_{k} \ldots r_{j} r_{j-1} \ldots r_{1}\right)\left(r_{1} \ldots r_{j-1}\right)=\left(r_{k} \ldots r_{1}\right)\left(r_{j-1} \ldots r_{1}\right)^{-1}= \\
& =\left(s_{1} \ldots s_{k}\right)\left(s_{1} \ldots s_{j-1}\right)^{-1}=\left(s_{1} \ldots s_{k}\right)\left(s_{j-1} \ldots s_{1}\right)=\left(s_{1} \ldots s_{j-1}\right) s_{j} \ldots s_{k}\left(s_{j-1} \ldots s_{1}\right),
\end{aligned}
$$

so we can compute

$$
\begin{aligned}
& r g=r_{j}\left(r_{k} \ldots r_{j+1} r_{j}\right)\left(r_{j-1} \ldots r_{1}\right)=r_{j}\left(s_{1} \ldots s_{j-1}\right) s_{j} \ldots s_{k}\left(s_{j-1} \ldots s_{1}\right)\left(r_{j-1} \ldots r_{1}\right)= \\
& =r_{j}\left(s_{1} \ldots s_{j-1}\right) s_{j} \ldots s_{k}\left(s_{j-1} \ldots s_{1}\right)\left(s_{1} \ldots s_{j-1}\right)=r_{j}\left(s_{1} \ldots s_{j-1}\right) s_{j} \ldots s_{k}= \\
& =\left(s_{1} \ldots s_{j-1}\right) s_{j}\left(s_{j-1} \ldots s_{1}\right)\left(s_{1} \ldots s_{j-1}\right) s_{j} \ldots s_{k}=\left(s_{1} \ldots s_{j-1}\right) s_{j} s_{j} \ldots s_{k}= \\
& =s_{1} \ldots s_{j-1} s_{j+1} \ldots s_{k}=s_{1} \ldots \hat{s}_{j} \ldots s_{k}
\end{aligned}
$$

15.5. ( $\star$ ) Let $(G, S)$ be Coxeter system such that its Coxeter diagram contains a cycle. Find an element of infinite order in $G$.

## Solution:

Let $1 \ldots k$ be a cycle. Consider an element $s_{1} \ldots s_{k}$. Then for any $m \in \mathbb{N}$ the element $\left(s_{1} \ldots s_{k}\right)^{m}$ is $M$-reduced: there is no elementary $M$-operation that can be applied to it. By Theorem 8.19, $\left(s_{1} \ldots s_{k}\right)^{m}$ is reduced, so $\left(s_{1} \ldots s_{k}\right)$ has infinite order.

